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CANADIAN JOURNAL OF MATHEMATICS

Journal Canadien de Mathématiques

VOL. I - NO. 2

1949

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Published for

THE CANADIAN MATHEMATICAL CONGRESS

by the University of Toronto Press

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The *Journal* is published quarterly. Subscriptions should be sent to the *Managing Editor*. The price per volume of four numbers is $6.00. This is reduced to $3.00 for individuals who are members of the following Societies:

Canadian Mathematical Congress
American Mathematical Society
Mathematical Association of America
London Mathematical Society
Société Mathématique de France

The Canadian Mathematical Congress gratefully acknowledges the assistance of the following towards the cost of publishing this Journal:

University of British Columbia
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MEAN-CONTINUOUS INTEGRALS

H. W. ELLIS

Introduction. Descriptive definitions of Cesàro-Denjoy integrals (CD-integrals) equivalent to the Cesàro-Perron integrals (CP-integrals) introduced by J. C. Burkill [1, 2] have been given by Miss Sargent [6] (see § 2). The CD-integrals are generalizations of the special Denjoy integral [5, p. 201]. They are somewhat complicated in that modifications of the definitions of continuity, generalized absolute continuity in the restricted sense (ACG_*) [5, p. 231], and of derivatives are required for each order. In the present paper a scale of integrals is obtained which is based on the descriptive definition of the general Denjoy integral [5, p. 241]. The approximate derivative and a slightly modified definition of generalized absolute continuity (ACG) are used for all orders so that the only concept generalized for increasing orders is that of continuity. The resulting r^{th} order integral, $r = 0, 1, 2, \ldots$, called the r^{th} generalized mean integral (GM_r-integral), contains the corresponding C_rD- and C_rP-integrals.

In § 1 the descriptive definition of the GM_r-integral is given and some of the more important properties of the integral, including a theorem on integration by parts, are derived. The relation between the GM_r-integral and the C_rD-, C_rP-integrals is considered in § 2. In § 3 a constructive definition of the GM_r-integral is given and shown to be equivalent to the descriptive definition. The paper concludes with a proof that the indefinite GM_r-integral takes all values between its upper and lower bounds on any interval over which the integral exists.

1. The descriptive generalized mean integrals. We shall obtain a scale or series of generalized mean integrals, GM_r-integrals, $r = 0, 1, \ldots$ of increasing generality in the sense that each integral will be contained in but not equivalent to all those with higher subscripts. We take as our starting point the general Denjoy integral.

Notation. We number theorems by Roman numerals, lemmas by Arabic numerals, and the definitions which change with the order by groups of letters. The order concerned in each case is indicated by a subscript, e.g. Definition (M_rC) is the definition of mean continuity of order r. With this notation we refer, for example, to "Theorem II$_r$" rather than "Theorem II for the GM_r-integral."

Received July 7, 1948. Most of the results of the present paper are adapted from a thesis accepted as partial fulfilment of the requirements for the degree of Doctor of Philosophy in the University of Toronto. The author wishes to express his thanks to Professor W. J. Webber for encouragement and suggestions in connection with the preparation of the thesis.

DEFINITION (M_r). *We define the M_r-mean of $F(x)$ on (a, b) as $F(b)$ for $r = 0$ and as*

$$M_r(F, a, b) = \frac{1}{(b-a)^r} \int_a^b (b-t)^{r-1} F(t)dt,$$

for r a positive integer, where the integral in the definition of the M_1-mean is in the general Denjoy sense and the sense in which the integral involved in the M_r-mean is required to exist will be stated below.

DEFINITION (M_rC). *The function $F(x)$ is continuous in the r^{th} mean sense (M_r-continuous) at x_0 if $M_r(F, x_0, x_0+h) \to F(x_0)$ as $h \to 0$.*

DEFINITION. *A function $F(x)$ is generalized absolutely continuous (ACG) on a set E if E can be expressed as the sum of a finite or denumerable sequence of closed sets $E_1, E_2, \ldots$ such that $F(x)$ is absolutely continuous (AC) on each set E_n.*[1]

DEFINITION (M_rI). *The function $f(x)$ is GM_r-integrable on (a, b) if there exists an M_r-continuous function $F(x)$ that is ACG on (a, b) and is such that the approximate derivative of $F(x)$, $ADF(x)$, [5, p. 220] exists and is equal to $f(x)$ almost everywhere on (a, b). The function $F(x)$ is then called an indefinite GM_r-integral of $f(x)$ on (a, b). The definite GM_r-integral of f over (a, b) is designated by*

$$GM_r(f, a, b) = (GM_r) \int_a^b f(x)dx = F(b) - F(a).$$

For $r = 0$ this definition is seen to be equivalent to the descriptive definition of the general Denjoy integral.

The integral in the definition of the M_r-mean, $r \geqq 2$, is required to exist in the sense of the GM_{r-1}-integral. In order to define the GM_r-integral we must therefore assume that the GM_{r-1}-integral has been defined. To establish the properties of the GM_r-integral we must assume the following properties for the GM_{r-1}-integral.

PROPERTY I_{r-1}. *If $f_1(x), f_2(x)$ are GM_{r-1}-integrable on (a, b) and $f_1(x) \geqq f_2(x)$ almost everywhere on (a, b) then*

$$GM_{r-1}(f_1, a, b) \geqq GM_{r-1}(f_2, a, b).$$

PROPERTY II_{r-1}. *The GM_{r-1}-integral contains the GM_{r-2}-integral.*

PROPERTY III_{r-1}. *Let $f_n(x)$ be GM_n-integrable on (a, b), let $F_n(x) = GM_n(f_n, a, x)$, $n = r-1, r-2, \ldots 0$, and let*

$$g_n(x) = \int_a^x dt_1 \int_a^{t_1} \ldots dt_{n-1} \int_a^{t_{n-1}} g(t)dt,$$

where $g(x)$ is of bounded variation on (a, b). Then $f_n(x)g_n(x)$ is GM_n-integrable on (a, b) and

$$(GM_n) \int_a^b f_n(x)g_n(x)dx = F_n(b)g_n(b) - (GM_{n-1}) \int_a^b F_n(x)g_{n-1}(x)dx.$$

---

[1] Saks' definition of ACG [5, p. 222] implies that $F(x)$ is continuous and does not require the sets E_n to be closed. The condition that the sets E_n be closed gives no restriction when $F(x)$ is continuous since the continuity of $F(x)$ is sufficient to ensure that if $F(x)$ is AC on an arbitrary set it is AC on the closure of this set.

(*Property* III_{r-1} *implies that, if* $f(x)$ *is* GM_{r-1}*-integrable, then* $f(x)g_{r-1}(x)$ *can be integrated by parts r times.*)

That properties I_{r-1}-III_{r-1} may be presumed is justified by induction. Since the GM_0-integral is the general Denjoy integral it is clear that Property I_0 is true and that Property III_0 is true provided that the GM_{-1}-integral is interpreted as a Stieltjes integral and $g(x)dx$ is replaced by $dg(x)$ [5, p. 246]. If $f(x)$ is GM_0-integrable on (a, b), $F(x) = GM_0(f, a, x)$ is ACG on (a, b) and $ADF(x) = f(x)$ almost everywhere on (a, b). To establish II_1 we need only prove that $F(x)$ is M_1-continuous and, since $F(x)$ is continuous, this is easily done. Our inductive process will be complete if, when we define the GM_r-integral and assume Properties I_{r-1}-III_{r-1}, we can establish Properties I_r-III_r.

LEMMA 1_r. *If there exists an interval* $(x_0, x_0 + h)$, $h > 0$ *and a positive number* d *such that* $F(t) - F(x_0) < d$ *for all* t *except at most a set of measure zero, then* $F(x)$ *cannot be* M_r*-continuous at* x_0.

Let x be any point in $(x_0, x_0 + h)$. Then, using Property I_{r-1},

$$\begin{aligned} M_r(F, x_0, x) &= r(x - x_0)^{-r}(GM_{r-1}) \int_{x_0}^{x} (x - t)^{r-1} F(t)\,dt \\ &\geqq r(x - x_0)^{-r} \int_{x_0}^{x} (x - t)^{r-1}[F(x_0) + d]\,dt \\ &= F(x_0) + d. \end{aligned}$$

It follows that $F(x)$ cannot be M_r-continuous at x_0. Similar results hold for $h < 0$ and also if $F(t) - F(x_0) > d$ is replaced by $F(t) - F(x_0) < -d$.

THEOREM I_r. *If* $F(x)$ *is monotone and* M_r*-continuous for* $a < x < b$ *then* $F(x)$ *is monotone and continuous for* $a \leqq x \leqq b$.

If we suppose that $F(x)$ is not continuous at some point x, $a \leqq x \leqq b$, Lemma 1_r gives a contradiction.

THEOREM II_r. *If* $F(x)$ *is* M_r*-continuous and* ACG *on* (a, b) *and if* $ADF(x) \geqq 0$ *almost everywhere on* (a, b), *then* $F(x)$ *is non-decreasing on* (a, b).

DEFINITION [4, p. 130]. *A function* $F(x)$ *is lower semi-absolutely continuous* ($\underline{AC}$) *over a set* E *if to a given positive number* ϵ *there corresponds a positive number* δ *such that for any non-overlapping set of intervals* (a_j, a'_j) *with* a_j, a'_j *points of* E, $\Sigma_j \{F(a'_j) - F(a_j)\} > -\epsilon$ *when* $\Sigma_j (a'_j - a_j) < \delta$.

DEFINITION. *A function* $F(x)$ *is generalized lower semi-absolutely continuous* ($\underline{ACG}$) *on* E *if* E *can be covered by a denumerable sequence of closed sets* $E_1, E_2, \ldots$ *such that* $F(x)$ *is* $\underline{AC}$ *on each set* E_i.

If "$> -\epsilon$" is replaced by "$< \epsilon$" in the above definitions, the corresponding definitions of upper and generalized upper semi-absolutely continuous ($\overline{AC}$) and ($\overline{ACG}$) functions are obtained. If $F(x)$ is ACG on (a, b) it is both $\underline{ACG}$ and $\overline{ACG}$ on (a, b).

LEMMA 2. *If $F(x)$ is $\underline{ACG}$ on (a, b) and if $ADF(x) \geqq 0$ almost everywhere on (a, b), then there is an interval (l, m) on (a, b) over which $F(x)$ is non-decreasing.*

Let $E_1, E_2, \ldots$ be the sets over which $F(x)$ is $\underline{ACG}$. Since these sets are closed there exists, by Baire's theorem [5, p. 54], an interval (l, m) on (a, b) such that $E_n(l, m) = (l, m)$ for some n.

Let G be the set of points of (l, m) at which $ADF \geqq 0$. For x a point of G we then have

$$\frac{F(x + h_i) - F(x)}{h_i} > -\epsilon,$$

with $x + h_i$ on a set of density unity at x, $h_i \to 0$. We can then use the Vitali covering theorem [5, p. 109] to get a finite non-overlapping set of intervals (x_k, x'_k) satisfying this relation and such that $\Sigma_k(x'_k - x_k) > mG - \delta = m - l - \delta$, where δ is sufficiently small to ensure that for (a_j, a'_j) a set of non-overlapping intervals with $\Sigma_j(a'_j - a_j) < \delta$ then

$$\Sigma_j\{F(a'_j) - F(a_j)\} > -\epsilon.$$

Let (a_j, a'_j) be the intervals complementary to the intervals (x_k, x'_k). Then

$$\begin{aligned} F(m) - F(l) &= \Sigma_k\{F(x'_k) - F(x_k)\} + \Sigma_j\{F(a'_j) - F(a_j)\} \\ &> -\epsilon(m - l) - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $F(m) - F(l) \geqq 0$.

In a similar manner it may be shown that $F(m') - F(l') \geqq 0$ for (l', m') any interval on (l, m). Hence $F(x)$ is non-decreasing on (l, m).

LEMMA 3_r. *Let $F(x)$ be M_r-continuous and $\underline{ACG}$ on (a, b) and let $ADF(x) \geqq 0$ almost everywhere on (a, b). If P is a perfect set on (a, b) with $F(x)$ non-decreasing on the intervals complementary to P, then there is an interval (l, m) containing points of P with $F(x)$ non-decreasing on (l, m).*

Since P is perfect, by Baire's theorem there exists an interval (l, m) containing points of P and such that $P(l, m)$ is identical with $E_k(l, m)$ for some k, where E_k is one of the sets over which $F(x)$ is $\underline{AC}$.

Let $F_1(x) = F(x)$ on $P(l, m)$ and let $F_1(x)$ be linear in the intervals complementary to $P(l, m)$ in such a way that $F_1(x)$ is continuous on (l, m). Then $F_1(x)$ is $\underline{AC}$ on (l, m). Since $F(x)$ is M_r-continuous and non-decreasing on an interval $a_j < x < b_j$ complementary to P it follows from Theorem I_r that $F(x)$ is continuous and non-decreasing on $a_j \leqq x \leqq b_j$. Hence $ADF_1(x) \geqq 0$ almost everywhere on (l, m). We can therefore use the argument of the preceding lemma to show that $F_1(x)$ is non-decreasing on (l, m) and $F(x)$ therefore non-decreasing on $P(l, m)$. We conclude that $F(x)$ is non-decreasing on (l, m).

THEOREM II'_r. *If $F(x)$ is M_r-continuous and $\underline{ACG}$ on (a, b) and if $ADF(x) \geqq 0$ almost everywhere on (a, b) then $F(x)$ is non-decreasing on (a, b).*

If we assume that Theorem II'_r is false we can use Lemmas 2 and 3_r as in

the proof of Theorem I, [4, p. 133], to obtain a contradiction. Theorem II_r then follows as a corollary.

THEOREM III_r. *If $f_1(x)$ and $f_2(x)$ are GM_r-integrable on (a, b) and $f_1(x) \geqq f_2(x)$ almost everywhere on (a, b) then $GM_r(f_1, a, b) \geqq GM_r(f_2, a, b)$.*

If we set $F_1(x) = GM_r(f_1, a, x)$ and $F_2(x) = GM_r(f_2, a, x)$, then $F_1(x) - F_2(x)$ is M_r-continuous and ACG on (a, b) and $AD[F_1(x) - F_2(x)] = ADF_1 - ADF_2(x) = f_1(x) - f_2(x) \geqq 0$ almost everywhere on (a, b). Hence, by Theorem II_r, $F_1(x) - F_2(x)$ is non-decreasing and, since $F_1(a) = F_2(a) = 0$, $F_1(b) \geqq F_2(b)$. We have therefore established Property I_r.

THEOREM IV_r. *If $f(x)$ is GM_{r-1}-integrable on (a, b) it is necessarily GM_r-integrable.*

We need only show that $F(x) = GM_{r-1}(f, a, x)$ is M_r-continuous. By hypothesis

$$(GM_{r-2}) \int_x^{x+h} (x + h - t)^{r-2} \{F(t) - F(x)\}\, dt = o(h^{r-1})$$

as $h \to 0$. The equality

$$\frac{r}{h^r} (GM_{r-2}) \int_x^{x+h} (x + h - t)^{r-1} \{F(t) - F(x)\}\, dt$$
$$= \frac{r(r-1)}{h^r} (GM_{r-2}) \int_x^{x+h} dt (GM_{r-3}) \int_x^t (t - \eta)^{r-2} \{F(\eta) - F(x)\}\, d\eta$$

may be established by integrating the integral on the left side by parts $r - 1$ times and the inner integral on the right $r - 2$ times [2, p. 543], operations which are justified by Property III_{r-1}. By Property II_{r-1} the integral on the left side exists in the GM_{r-1}-sense and has the same value. The right hand side is equal to

$$r(r-1)h^{-r} \int_x^{x+h} o\,[(t - x)^{r-1}]\, dt = o(1)$$

as $h \to 0$. We have therefore established Property II_r.

THEOREM V_r. *Let $F(x)$ be M_r-continuous at x and let*

$$g_r(t) = \int_a^t dt_1 \int_a^{t_1} \ldots dt_{r-1} \int_a^{t_{r-1}} g(t) dt,$$

where $g(t)$ is of bounded variation on (a, x). Then $F(x)g_r(x)$ is M_r-continuous at x.

Using the Cesàro-Perron analogue of Property III_{r-1} Burkill [2, p. 549] establishes Theorem V_r for the C_rP-integral. The method of proof applies as well to the GM_r-integral.

THEOREM VI_r (Integration by parts). *Let $f(x)$ be GM_r-integrable on (a, b) and let $g_r(x)$ be defined as in Theorem V_r. Then $f(x)g_r(x)$ is GM_r-integrable on (a, b) and*

$$(GM_r) \int_a^b f(x)g_r(x) = F(b)g_r(b) - (GM_{r-1}) \int_a^b F(x)g_{r-1}(x) dx.$$

By Property III_{r-1} $F(x)g_{r-1}(x)$ is GM_{r-1}-integrable. We set

$$H(x) = F(x)g_r(x) - (GM_{r-1})\int_a^x F(t)g_{r-1}(t)dt.$$

By Theorem V_r $F(x)g_r(x)$ is M_r-continuous on (a, b). By Theorem IV_r $F(t)g_{r-1}(t)$ is also GM_r-integrable and to $GM_{r-1}(Fg_{r-1}, a, x)$. Hence $GM_{r-1}(Fg_{r-1}, a, x)$ is M_r-continuous on (a, b). Since the sum of two M_r-continuous functions is M_r-continuous, $H(x)$ is M_r-continuous.

Now $F(x)$ is ACG and $g_r(x)$ is AC on (a, b). Since the product of two AC functions is AC, $F(x)g_r(x)$ is AC over each of the closed sets over which $F(x)$ is AC and therefore $F(x)g_r(x)$ is ACG on (a, b). Since $GM_{r-1}(Fg_{r-1}, a, x)$ is also ACG on (a, b) it follows that $H(x)$ is ACG on (a, b).

As GM-integrals the functions $F(x)$ and $GM_{r-1}(Fg_{r-1}, a, x)$ are approximately derivable almost everywhere on (a, b). Since $ADF(x) = f(x)$ almost everywhere on (a, b), and $AD\,[GM_{r-1}(Fg_{r-1}, a, x)]$ is equal to $F(x)g_{r-1}(x)$ almost everywhere on (a, b), $ADH(x) = ADF(x)g_r(x) = f(x)g_r(x)$ almost everywhere on (a, b).

It follows that $H(x)$ is an indefinite GM_r-integral of $f(x)g_r(x)$ and

$$(GM_r)\int_a^b f(x)g_r(x)dx = H(b) = F(b)g_r(b) - (GM_{r-1})\int_a^b F(x)g_{r-1}(x)dx.$$

We have therefore proved Theorem VI_r and established Properties I-III by induction. Property II shows that the GM-scale of integrals is consistent, i.e. that each integral contains those with lower subscripts. Each integral of the scale is more general than the preceding integral. This is shown by the example $f(x) = 0$, $x = 0$, $f(x) = (d/dx)^p x^q \sin(1/x)$, p, q integers $x \neq 0$ for, given r, $f(x)$ will be a GM_r-integrable but not GM_{r-1}-integrable function if p and q are properly chosen.

A further property of the GM_r-integral that follows easily from Theorem II_r is the following.

The definite GM_r-integral is determined uniquely, and the indefinite GM_r-integral is determined uniquely apart from an additive constant.

We list other properties for which the proofs are essentially the same for higher orders as for the GM_0- or general Denjoy integral.

A function which is GM_r-integrable is necessarily measurable and almost everywhere finite [5, p. 243].

The function $F(x) = GM_r(f, a, x)$, $a \leqq x \leqq b$, satisfies Lusin's condition (N) [5, p. 225].

A function which is GM_r-integrable and almost everywhere non-negative on an interval (a, b) is necessarily Lebesgue integrable on (a, b) [5, p. 242].

Given a non-decreasing sequence of functions $f_n(x)$ which are GM_r-integrable on an interval (a, b) and whose GM_r-integrals over (a, b) constitute a sequence bounded above, then the function $f(x) = \lim f_n(x)$ is itself necessarily GM_r-integrable on (a, b) and

$$(GM_r)\int_a^b f(x)dx = \lim_{n\to\infty}(GM_r)\int_a^b f_n(x)dx,$$

[5, p. 243].

2. The relation between the GM_r- and the C_rP-, C_rD-integrals. The C_0P-integral is the Perron integral and the C_rP-integral is defined in an analogous manner [1, 2] using major and minor functions but with ordinary continuity and derivatives replaced by mean (Cesàro) continuity and derivatives.

DEFINITION (AC^*_r) [6, p. 221]. *The function $F(x)$ is said to be AC^* (C_r-sense) over a set E if it is $C_{r-1}D$-integrable in an interval which contains E and if to each positive number ϵ corresponds a number δ such that*[2]

$$\sum_{i=1}^{n} \overline{\text{bound}}_{a_i<x<b_i} \left|C_r(F, a_i, x) - F(a_i)\right| < \epsilon,$$

$$\sum_{i=1}^{n} \overline{\text{bound}}_{a_i<x<b_i} \left|C_r(F, b_i, x) - F(b_i)\right| < \epsilon,$$

for all finite sets of non-overlapping intervals (a_i, b_i) with end points on E and such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

DEFINITION (ACG^*_r) [6, p. 221]. *The function $F(x)$ will be said to be ACG^* (C_r-sense) over a set E if $F(x)$ is C_r-continuous at points of E and if E is the sum of a denumerable number of sets over each of which $F(x)$ is AC^*(C_r-sense).*

DEFINITION (DI_r) [6, p. 232]. *The function $f(x)$ is said to be C_rD-integrable in (a, b) if there is a function $F(x)$ that is ACG^*(C_r-sense) over the closed interval (a, b) and such that $C_rDF(x) = f(x)$ almost everywhere in (a, b). Then $F(x)$ is an indefinite C_rD-integral of $f(x)$, $F(b) - F(a)$ the definite C_rD-integral in (a, b).*

By basing our generalizations on the general Denjoy integral we were able to obtain a considerably simpler descriptive definition of an r^{th} order integral. The condition that $F(x)$ be C_r-continuous could be separated from Definition (ACG^*_r) and included separately in Definition (DI_r). The concept of (ACG) is then seen to be simpler than the modified Definition (ACG^*_r). Furthermore, in Definition (DI_r) the concepts of both ACG^*(C_r-sense) and C_r-derivatives must be modified for each r. We prove that the simpler GM_r-integral is actually more general than the C_rD- and equivalent C_rP-integrals.

THEOREM VII$_r$. *The GM_r-integral contains the C_rD- and C_rP-integrals.*

Since Miss Sargent [6] proved the equivalence of the C_rD- and C_rP-integrals we need only show that the GM_r-integral contains the C_rD-integral. We proceed by induction. Since it is well known that the GM_0-integral contains the C_0D- or special Denjoy integral we may suppose the theorem true for orders less than r and prove that it is then true for order r.

---

[2] The r^{th} Cesàro mean of F on (a, b) is denoted by $C_r(F, a, b)$ and differs from the M_r-mean only in that the integral is required to exist in the C_rD- sense rather than the GM_r-sense.

We suppose that f is C_rD-integrable and set $F(x) = C_rD(f, a, x)$. Then, since $F(x)$ is C_r-continuous,

$$\frac{r}{h^r}(C_{r-1}D)\int_a^{x+h}(x+h-t)^{r-1}F(t)dt \to F(x)$$

as $h \to 0$. By hypothesis the integral exists in the GM_{r-1}-sense and has the same value. We conclude that $F(x)$ is M_r-continuous on (a, b).

By the descriptive definition of the C_rD-integral, (a, b) can be covered by a sequence of closed sets (E_n) over each of which $F(x)$ is $AC^*(C_r$-sense). By Theorem II [6, p. 227] a necessary condition for $F(x)$ to be $AC^*(C_r$-sense) over a set E_n is that $F(x)$ be AC on E_n. It follows that $F(x)$ is ACG on (a, b).

In [6, p. 228] it is shown that if $F(x)$ is $AC^*(C_r$-sense) on a set E_n, then the C_r-derivative [2, p. 542] $C_rDF(x)$ exists, is finite and equal to $ADF(x)$ at almost all points x of E_n. Since (a, b) is covered by at most a denumerable sequence of such sets it follows that $C_rDF(x) = ADF(x)$ almost everywhere on (a, b). Since, by the definition of a C_rD-integral, $C_rDF(x) = f(x)$ almost everywhere it follows that $ADF(x) = f(x)$ almost everywhere on (a, b). It then follows that $f(x)$ is GM_r-integrable to $F(x)$.

On the other hand the C_rP-, C_rD-integrals do not contain the GM_r-integral. This is well known for $r = 0$ since the special Denjoy integral is contained in but not equivalent to the general Denjoy integral. A similar relation holds for other values of r. We therefore have two distinct scales: (1) the CD-, CP-scale of integrals similar to and generalizing the Denjoy-Perron integral; and (2) the GM-scale of integrals similar to and generalizing the general Denjoy integral and such that the GM_r-integral contains the C_rD-, C_rP-integrals.

3. The constructive GM_r-integral. To obtain a constructive definition of the GM_r-integral we modify the definitions and conditions for integrability in the general Denjoy sense by using limits involving M_r-means.

DEFINITION (a). *If the function $f(x)$ is summable over a measurable set E then $GM_r(f, E)$ is $L(f, E)$ the Lebesgue integral of $f(x)$ over E.*

DEFINITION (b). *Let (α_i, β_i) be any interval and suppose that $GM_r(f, \alpha, \beta)$ has been determined for every interval (α, β) interior to (α_i, β_i). Let ξ be a point with $\alpha_i < \xi < \beta_i$ and let $F(t) = GM_r(f, t, \xi)$. Let $K_r(\alpha_i, \xi)$ and $K_r(\xi, \beta_i)$ be the respective limits as $h \to 0^+$ of*

$$rh^{-r}(GM_{r-1})\int_{\alpha_i}^{\alpha_i+h}(\alpha_i+h-t)^{r-1}F(t)dt,$$

$$rh^{-r}(GM_{r-1})\int_{\beta_i-h}^{\beta_i}(t-\beta_i+h)^{r-1}F(t)dt,$$

where the integrals are supposed to tend to limits which are finite. If $f(x)$ is such that $K_r(\alpha_i, \xi) + K_r(\xi, \beta_i)$ is independent of ξ then

$$GM_r(f, \alpha_i, \beta_i) = K_r(\alpha_i, \xi) + K_r(\xi, \beta_i).$$

DEFINITION (c). *Let E be a closed set over which $f(x)$ is summable, (α_i, β_i) the intervals complementary to E on (a, b); suppose that $GM_r(f, \alpha_i, \beta_i)$ has been determined for all the intervals (α_i, β_i), and that (l, m) is an interval for which $\sum_{(l,m)} |GM_r(f, \alpha_i, \beta_i)|$ converges. Then*

$$GM_r(f, l, m) = \int_{E(l,m)} f(x)dx + \sum_{(l,m)} GM_r(f, \alpha_i, \beta_i),$$

where it is understood that if l is interior to an interval (α_k, β_k) then the term in the sum arising from this interval is $GM_r(f, l, \beta_k)$. A similar understanding holds if m is not a point of E.

The function $f(x)$ is then GM_r-integrable on (a, b) if it satisfies the following conditions:

(1) If E is any closed set on (a, b) there exists an interval (l, m) containing points of E and such that $f(x)$ is summable over $E(l, m)$.

(2) The function $f(x)$ is such that the limits in (b) exist.

(3) If $GM_r(f, l, x)$ exists for all x in an interval (l, m) then it is M_r-continuous as a function of x in (l, m).

(4) The function $f(x)$ is such that if E is any closed set for which $GM_r(f, \alpha_i, \beta_i)$ has been determined for all intervals (α_i, β_i) complementary to E, there exists an interval (l, m) containing points of E and such that $\sum_{(l,m)} |GM_r(f, \alpha_i, \beta_i)|$ converges.

Definitions (a), (b) and (c) together with conditions (1), (2), and (4) permit the determination of $GM_r(f, a, b)$ in a finite or denumerable number of steps as in [3, p. 20 ff.]. Further conditions are needed to ensure that $F(x) = GM_r(f, a, x)$ is M_r-continuous. These conditions are discussed for an integral equivalent to the GM_1-integral in [3]. We have postulated mean continuity by adding condition (3).

If we set $F(a) = 0$, $F(x) = GM_r(f, a, x)$ for $a < x \leqq b$, we can prove Lemmas 4_r and 5_r as in [3, Theorems I and II].

LEMMA 4_r. *The function $F(x) = GM_r(f, a, x)$ is ACG on (a, b).*

LEMMA 5_r. *At almost all points of (a, b) $ADF(x)$ exists and is equal to $f(x)$.*

THEOREM VIII$_r$. *The constructive and descriptive definitions of the GM_r-integral are equivalent.*

Condition (3) and Lemmas 4_r and 5_r show that the descriptive integral contains the constructive integral. We must therefore show that if the M_r-continuous function $F(x)$ is ACG on (a, b) and such that $ADF(x)$ is finite almost everywhere and equal to $f(x)$, then the constructive definition gives $GM_r(f, a, x) = F(x) - F(a)$. We first prove a lemma.

LEMMA 6_r. *Let $F(x)$ be ACG on (a, b) and let $ADF(x)$ be finite and equal to $f(x)$ almost everywhere on (a, b). If E is any closed set on (a, b) there then exists an interval (l, m) such that $f(x)$ is summable over $E(l, m)$, $\sum_{(l,m)} |F(\beta_i) - F(\alpha_i)|$*

converges, where (α_i, β_i) *are the intervals complementary to* $E(l, m)$, *and for any such interval*

$$F(m) - F(l) = \int_{E(l,m)} f(x)dx = \sum_{(l,m)} \{F(\beta_i) - F(\alpha_i)\}.$$

Let $A_1, A_2, \ldots$ be the closed sets over which $F(x)$ is AC. There then exists, by Baire's Theorem, an interval (l, m) and an integer k such that A_k and E are identical on (l, m). Since F is AC on A_k, $D_{A_k}F(x)$ exists for almost all points of A_k and is summable over A_k. Then, since for almost all points of A_k we have $D_{A_k}F = ADF = f$, it follows that f is summable over $E(l, m)$. The convergence of $\sum_{(l,m)} |F(\beta_i) - F(\alpha_i)|$ follows from the absolute continuity of F on $E(l, m)$.

Let $G(x) = F(x)$ on $E(l, m)$ and be linear in the intervals (α_i, β_i) in such a way as to be continuous on (l, m). Then $G(x)$ is AC on (l, m) and, at almost all points of $E(l, m)$, $G' = D_{E(l,m)}F = ADF = f$. Hence

$$\begin{aligned} F(m) - F(l) &= (L)\int_l^m G'(x)dx \\ &= \int_{E(l,m)} f(x)dx + \sum_{(l,m)} \int_{\alpha_i}^{\beta_i} G'(x)dx \\ &= \int_{E(l,m)} f(x)dx + \sum_{(l,m)} \{F(\beta_i) - F(\alpha_i)\}. \end{aligned}$$

We return to the proof of the theorem and, as in the existence proof for the constructive GM_r-integral [3, pp. 21-23], we let E_1 be the points of non-summability of F on (a, b). Then E_1 is closed. If we denote by (α_i^1, β_i^1) the intervals complementary to E_1, by (α, β) an interval with $\alpha_i^1 < \alpha < \beta < \beta_i^1$, then

$$F(\beta) - F(\alpha) = L(f, \alpha, \beta) = GM_r(f, \alpha, \beta).$$

Since $F(x)$ is M_r-continuous, $F(\beta) - F(\alpha)$ tends in the M_r-sense to the limit $F(\beta_i^1) - F(\alpha_i^1)$ which is finite and independent of any ξ, $\alpha_i^1 < \xi < \beta_i^1$. It follows that Definition (b) applies to $GM_r(f, \alpha, \beta)$ and

$$F(\beta_i^1) - F(\alpha_i^1) = \lim_{\substack{\alpha \to \alpha_i^1 \\ \beta \to \beta_i^1}} GM_r(f, \alpha, \beta) = GM_r(f, \alpha_i^1, \beta_i^1).$$

By Lemma 6_r there exists at least one interval (l, m) containing points of E_1 and such that f is summable on $E_1(l, m)$ and $\sum_{(l,m)} |F(\beta_i^1) - F(\alpha_i^1)|$ converges. Let E_2 be the points of E_1 that are points of non-summability of f over E_1 and/or points x such that $\sum_{(l,m)} |F(\beta_i^1) - F(\alpha_i^1)|$ diverges for every interval (l, m) containing x. If (α_i^2, β_i^2) are the intervals complementary to E_2 and (α, β) is an interval with $\alpha_i^2 < \alpha < \beta < \beta_i^2$ then, by Lemma 6_r,

$$F(\beta) - F(\alpha) = \int_{E_1(\alpha,\beta)} f(x)dx + \sum_{(\alpha,\beta)} \{F(\beta_i^1) - F(\alpha_i^1)\},$$

and the right side is now equal to $GM_r(f, \alpha, \beta)$ by Definition (c). As before

we can use the M_r-continuity of F to determine $GM_r(f, \alpha_i^2, \beta_i^2) = F(\beta_i^2) - F(\beta_i^2)$. Continuing this process we can determine $GM_r(f, l, m) = F(m) - F(l)$ in a finite or denumerable number of steps.

4. A continuity property of GM_r-integrals. We conclude with a proof that an important property of continuous functions extends to mean continuous indefinite integrals. This result has been stated for the C_1P-integral [7, p. 238].

THEOREM IX$_r$. *If $F(x)$ is an indefinite GM_r-integral of $f(x)$ defined on (a, b) and if (l, m) is any closed interval on (a, b), then $F(x)$ takes all values between its upper and lower bounds on (l, m) for $l \leqq x \leqq m$.*

Let E_1 be the points of non-summability of f on (l, m). If (α_i^1, β_i^1) are the intervals complementary to E_1 on (l, m), α_i^1, β_i^1 points of E_1 and (α, β) is an interval with $\alpha_i^1 < \alpha < \beta < \beta_i^1$, then f is Lebesgue integrable on (α, β). It follows that $F(x)$ takes all values between its upper and lower bounds on (α, β) for $\alpha \leqq x \leqq \beta$.

Let β be fixed, let α tend to α_i^1 and consider the intervals (α_i^1, α). There are three possibilities: (i) Every interval (α_i^1, α) contains points x with $F(x) > F(\alpha_i^1)$ and points x' with $F(x') < F(\alpha_i^1)$; (ii) There exists an interval (α_i^1, α) with no point x such that $F(x) > F(\alpha_i^1)$; or (iii) There exists an interval (α_i^1, α) containing no point x such that $F(x) < F(\alpha_i^1)$.

In the first case it is clear that $F(x)$ takes all values between its upper and lower bounds on (α_i^1, β) for $\alpha_i^1 \leqq x \leqq \beta$. In the second case, given an arbitrary $\epsilon > 0$ there exists δ such that $F(\alpha_i^1) - F(x_1) < \epsilon$ for some x_1, $\alpha_i^1 < x_1 < \alpha_i^1 + \delta$. If not, Lemma 1$_r$ would contradict the M_r-continuity of $F(x)$ at α_i^1. Since $F(x)$ takes all values between its upper and lower bounds on (x_1, β) and ϵ is arbitrary, it follows that it takes all values between its upper and lower bounds on (α_i^1, β). Since a similar argument holds for β tending to β_i^1, $F(x)$ takes all values between its upper and lower bounds on (α_i^1, β_i^1) for $\alpha_i^1 \leqq x \leqq \beta_i^1$. A similar argument holds if (iii) applies.

As in the transfinite process by which the GM_r-integral was built up from the constructive definition [3, p. 20], let E_2 be the points x of E_1 such that one or both of the following conditions hold: (i) For every interval (c, d) containing x the function f is not summable over $E_1(c, d)$; (ii) The sum $\sum_{(c,d)} |GM_r(f, \alpha_i^1, \beta_i^1)|$ diverges for every interval (c, d) containing x. Let (α_i^2, β_i^2) be an interval of the set complementary to E_2 with α_i^2, β_i^2 points of E_2; (α, β) an interval with $\alpha_i^2 < \alpha < \beta < \beta_i^2$.

Let $G(x) = F(x)$ for $x = \alpha, \beta$ and at points of $E_1(\alpha, \beta)$ and let $G(x)$ be linear in the intervals (α_i^1, β_i^1) on (α, β) and in the intervals (α, α_k^1), $(\beta_{k'}^1, \beta)$ where α_k^1, $\beta_{k'}^1$, are the upper and lower bounds of points of E_1 on (α, β).

Since α, β are arbitrary it is sufficient to prove that $F(x)$ takes all values between $F(\alpha)$ and $F(\beta)$ on (α, β). Let c be any value between $F(\alpha)$, $F(\beta)$. Then $G(x') = c$ for some x', $\alpha < x' < \beta$. If $x' \epsilon E_1$, then $F(x') = c$. If $G(x) \neq c$ in E_1 there exists some interval (α_j^1, β_j^1) with $\alpha_j^1 < x' < \beta_j^1$ and $F(\alpha_j^1) = G(\alpha_j^1) <$

$c < G(\beta_j^1) = F(\beta_j^1)$ or the reverse inequality. By the first part of the theorem $F(x)$ takes all values between $F(\alpha_j^1)$ and $F(\beta_j^1)$ for $\alpha_j^1 < x < \beta_j^1$ and therefore takes the value c.

As before we can pass from the intervals (α, β) to the intervals (α_i^2, β_i^2) complementary to E_2. Continuing this process we can establish the theorem in a finite or denumerable number of steps.

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The University of Toronto and Queen's University

COMPLETELY INDECOMPOSABLE MODULES

ERNST SNAPPER

Introduction and summary. The purpose of this paper is to investigate completely indecomposable modules. A completely indecomposable module is an additive abelian group $\mathfrak{B}$ with a ring A as operator domain, where the following four conditions are satisfied.

I-1. A is a commutative ring and has a unit element which is unit operator for $\mathfrak{B}$.

I-2. The submodules of $\mathfrak{B}$ satisfy the ascending chain condition. (Submodule will always mean invariant submodule.)

I-3. The submodules of $\mathfrak{B}$ satisfy the descending chain condition.

I-4. Every submodule of $\mathfrak{B}$ ($\mathfrak{B}$ itself included) is indecomposable.

Properties I-1 and I-2 are equivalent with saying that $\mathfrak{B}$ is a Noetherian module. (See [1] and [2] for Noetherian modules; square brackets refer to the references.) Property I-4 says that if $\mathfrak{W}$ is a submodule of $\mathfrak{B}$, then $\mathfrak{W}$ is not decomposable in a direct sum $\mathfrak{W} = \mathfrak{W}_1 \dot{+} \mathfrak{W}_2$, where $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are submodules of $\mathfrak{W}$. (The symbol $\dot{+}$ will always denote a *direct* sum.) It is easily shown that I-4 is equivalent with saying that $\mathfrak{B}$ has a unique minimal submodule (see remark 1.1).

The notion of a completely indecomposable module arises in a natural way from the study of the classical elementary divisor theory. As is shown in sec. 6, every indecomposable module which occurs in the classical elementary divisor theory and whose annihilating ideal is not the zero ideal (i.e., for whose submodules the descending chain condition holds) is completely indecomposable.

The main result of this paper is Theorem 5.1. It states that *two completely indecomposable modules are isomorphic* (isomorphic will always mean operator-isomorphic) *if and only if they have the same annihilating ideal.* In the language of ring-representations this means: *two faithful representations of a commutative ring with unit element* 1, *whose respective representation spaces have composition sequences and unique minimal sub-spaces, while* 1 *is unit operator, are equivalent.* (See remark 5.1.) Theorem 5.1 can immediately be extended to Theorem 5.2 on *regular modules* (i.e., modules which are direct sums of completely indecomposable modules; see definition 5.1.) Thus we obtain a generalization to regular modules of that part of the classical elementary divisor theory which is concerned with modules in which the descending chain condition is satisfied (see sec. 6).

Sec. 1 contains that part of the theory of a completely indecomposable module $\mathfrak{B}$ which follows from the theory of Noetherian modules. Sec. 2 con-

Received January 26 and October 25, 1948.

tains the part of the theory of $\mathfrak{B}$ which is obtained by extending Gröbner's theory of irreducible ideals (see [3]) to completely indecomposable modules. Sec. 3 contains the proof that the operator-endomorphism ring of $\mathfrak{B}$ consists of the multiplications of the elements of $\mathfrak{B}$ with the elements of the operator domain A. This does not imply that $\mathfrak{B}$ is necessarily cyclic (see remark 3.1). Lemma 3.1, which is concerned with the extension of operator-isomorphisms, holds for abelian groups with arbitrary rings, not necessarily commutative, as operator domain. Sec. 4 contains two known lemmas of ring-representation theory which are needed for secs. 5 and 6. Sec. 5 contains the proof of the main theorem 5.1. In sec. 6, it is shown how that part of the classical elementary divisor theory which is concerned with modules in which the descending chain condition is satisfied, Steinitz theory of algebraic integers included, can be obtained from the theory of completely indecomposable modules. Here, the main lemma is lemma 6.1, which states that *an indecomposable module which satisfies conditions* I-1, I-2 *and* I-3 *and whose annihilating ideal is intersection-irreducible is completely indecomposable and cyclic.* Sec. 7 contains examples of completely indecomposable modules. The unsolved problems to which this paper gives rise are stated in remarks 6.2 and 7.2.

It is pointed out in remark 7.1 that the theory of completely indecomposable modules gives rise to the question "does every commutative, completely primary ring have a faithful, completely indecomposable representation space?" This question is answered in the affirmative by corollary 9.4 of sec. 9. In order to prove this corollary it was necessary to introduce in sec. 8 a new composition of operator modules, called *interlacing of modules*, which is a generalization of the direct sum. *Sec. 8 is self-contained and demonstrates that interlacing can be used equally well for non-commutative as for commutative operator domains. The author believes that many other uses can be made of the notion of interlacing.* Finally remark 9.2 shows how, as a consequence of corollary 9.4, we can define a "dual vector space" for any commutative ring A with unit element whose ideals satisfy both chain conditions. This dual vector space is a generalization of the dual space defined in [8], p. 558, for the case where A is an algebra of finite rank with respect to a field.

The author is indebted to Professors R. Brauer and R. M. Thrall for valuable help in the writing of this paper. (See secs. 4 and 6 and example 7.3 and remark 7.2.)

1. Properties of completely indecomposable modules obtained from the theory of Noetherian modules. Let $\mathfrak{B}$ be a completely indecomposable module with the ring A as right-operator domain. Hence the conditions I-1, I-2, I-3 and I-4 of the introduction are satisfied. The elements of $\mathfrak{B}$ will be denoted by capital Latin letters and the submodules of $\mathfrak{B}$ by capital German letters. The elements of A will be denoted by lower case Greek letters, and the ideals of A by lower case German letters.

Since $\mathfrak{B}$ is a Noetherian module (see [2], sec. 2.1, for the definition of Noe-

therian module; in [2] the term Noetherian vector space is used) which has a composition series of finite length (this follows from I-2 and I-3), the associated primes of the zero-module 0 of $\mathfrak{B}$ are maximal prime ideals of A. (For the definition of associated primes of a module, see [2], Theorem 2.21. The fact that the associated primes of 0 have to be maximal is stated in Theorem 24 of [1].) Since $\mathfrak{B}$ is itself indecomposable, 0 can have only one associated prime $\mathfrak{p}$, namely the radical of the annihilating ideal $\mathfrak{q}$ of $\mathfrak{B}$. (Since all the associated primes of 0 are maximal, $\mathfrak{B}$ would be decomposable if 0 had more than one associated prime; this follows immediately from Theorem 19 of [1].) Consequently, 0 is a primary submodule of $\mathfrak{B}$ whose fundamental ideal is the annihilating ideal $\mathfrak{q}$ of $\mathfrak{B}$ and whose radical is the maximal ideal $\mathfrak{p}$. (See sec. 2.1 of [2] for the definitions of primary module, fundamental ideal and radical.) This implies that $\mathfrak{q}$ is primary (see Theorem 2.11 of [2]) and that if $V\mathfrak{a} = 0$, where $V \in \mathfrak{B}$ and $V \neq 0$ and $\mathfrak{a} \in A$, then $\mathfrak{a} \in \mathfrak{p}$ (see definition 2.12 of [2]).

Now, let $0 = \mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \cdots \subset \mathfrak{B}_{l-1} \subset \mathfrak{B}_l = \mathfrak{B}$ be a composition series of $\mathfrak{B}$. (The symbol $\subset$ will be used exclusively for *proper* inclusion). Since the fundamental ideal of 0 is primary and has maximal radical, the difference module $\mathfrak{B}_i - \mathfrak{B}_{i-1}$ is operator-isomorphic with $A - \mathfrak{p}$; hence the quotient $\mathfrak{B}_{i-1} : \mathfrak{B}_i = \mathfrak{p}$ for $i = 1, \ldots, l$. (The quotient $\mathfrak{B}_{i-1} : \mathfrak{B}_i$ is defined as the ideal which consists of all $\mathfrak{a} \in A$, such that $V\mathfrak{a} \in \mathfrak{B}_{i-1}$ for all $V \in \mathfrak{B}_i$; see sec. 2.1 of [2]. The fact that the difference module $\mathfrak{B}_i - \mathfrak{B}_{i-1}$ is isomorphic with $A - \mathfrak{p}$ is stated in Theorem 2.41 of [2].) In particular, $\mathfrak{B}_1\mathfrak{p} = 0$, which implies that $\mathfrak{B}_1 \subseteq 0 : \mathfrak{p}$. (See sec. 2.1 of [2] for the definitions of the product of a module with an ideal and of the quotient of a module by an ideal.) However, if $0 : \mathfrak{p}$ were not a minimal module of $\mathfrak{B}$ (that is, a module which contains 0 as only proper submodule), then $0 : \mathfrak{p}$ would be decomposable in a direct sum of cyclic submodules (see [1], sec. 16). Consequently, property I-4 implies that $\mathfrak{B}_1 = 0 : \mathfrak{p}$, which, since $\mathfrak{B}_1$ is an arbitrary minimal submodule of $\mathfrak{B}$, proves the following lemma.

LEMMA 1.1. *A completely indecomposable module $\mathfrak{B}$ has a unique minimal submodule, namely* $0 : \mathfrak{p}$.

REMARK 1.1. If $\mathfrak{B}$ satisfies properties I-1, I-2 and I-3, then I-4 is obviously equivalent with the existence of a unique minimal submodule in $\mathfrak{B}$. In the first place, if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ were two distinct minimal submodules, the submodule $\mathfrak{W} = \mathfrak{B}_1 \dotplus \mathfrak{B}_2$ would be decomposable. Conversely, if $\mathfrak{B}_1$ is a unique minimal submodule of $\mathfrak{B}$, then $\mathfrak{B}_1$ is contained in every submodule of $\mathfrak{B}$, which implies that $\mathfrak{B}$ is completely indecomposable. The main content of lemma 1.1 is the fact that this unique submodule is $0 : \mathfrak{p}$.

REMARK 1.2. It is important for secs. 2, 5 and 6 to observe that property I-4 is also equivalent with saying that 0 is an intersection-irreducible module (i.e., a module which is not the intersection of two proper divisors). In the first place, if $0 = \mathfrak{W}_1 \cap \mathfrak{W}_2$, where $0 \subset \mathfrak{W}_1$ and $0 \subset \mathfrak{W}_2$, then the sum $(\mathfrak{W}_1, \mathfrak{W}_2)$ of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ is decomposable, $(\mathfrak{W}_1, \mathfrak{W}_2) = \mathfrak{W}_1 \dotplus \mathfrak{W}_2$, and hence $\mathfrak{B}$ is then not completely indecomposable. Conversely, if $\mathfrak{B}$ is not completely indecom-

posable, $\mathfrak{B}$ contains a decomposable module $\mathfrak{W} = \mathfrak{W}_1 \dot{+} \mathfrak{W}_2$, where $0 \subset \mathfrak{W}_1$ and $0 \subset \mathfrak{W}_2$. Then $0 = \mathfrak{W}_1 \subset \mathfrak{W}_2$ and hence 0 is then not intersection-irreducible.

2. Properties of completely indecomposable modules obtained from Gröbner's theory of irreducible ideals. Gröbner investigated in [3] the theory of an intersection-irreducible ideal $\mathfrak{q}$ of a commutative ring A, where the ideals of A satisfy the ascending chain condition. Since an ideal is intersection-irreducible if it is not the intersection of two proper divisors, the zero module of the difference module $A - \mathfrak{q}$ (difference modules are always considered as modules with A as operator domain) is also intersection-irreducible. Since the zero module of a completely indecomposable module is intersection-irreducible (see remark 1.2), Gröbner's theory, as was shown by Grundy in [1], sec. 16, can be extended to completely indecomposable modules. The following lemmas 2.1 and 2.2, which will be used in secs. 3 and 5, are proved in [1], sec. 16.

Let $\mathfrak{B}$ again be a completely indecomposable module with A as operator domain and $\mathfrak{q}$ as annihilating ideal; and let $\mathfrak{p}$ again be the radical of 0. Since $\mathfrak{q}$ is a primary ideal of A with maximal radical $\mathfrak{p}$ (see sec. 1), every divisor of $\mathfrak{q}$, different from A, is a primary ideal with $\mathfrak{p}$ as radical. Furthermore, according to [2], Theorem 2.41, there exists a composition series $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_{l'-1} \subset \mathfrak{q}_{l'} = A$ from $\mathfrak{q}$ to A, where the difference module $\mathfrak{q}_i - \mathfrak{q}_{i-1}$ is operator isomorphic with $A - \mathfrak{p}$ and where $\mathfrak{q}_{i-1} : \mathfrak{q}_i = \mathfrak{p}$ for $i = 1, \ldots, l'$.

LEMMA 2.1. *If $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_{l'-1} \subset \mathfrak{q}_{l'} = A$ is a composition series from $\mathfrak{q}$ to A, then $0 \subset 0 : \mathfrak{q}_{l'-1} \subset \ldots \subset 0 : \mathfrak{q}_1 \subset \mathfrak{B}$ is a composition series of $\mathfrak{B}$. Conversely, if $0 \subset \mathfrak{B}_1 \subset \ldots \subset \mathfrak{B}_{l-1} \subset \mathfrak{B}$ is a composition series of $\mathfrak{B}$, then $\mathfrak{q} \subset 0 : \mathfrak{B}_{l-1} \subset \ldots \subset 0 : \mathfrak{B}_1 \subset A$ is a composition series from $\mathfrak{q}$ to A. Consequently, the length l' of the difference module $A - \mathfrak{q}$ is equal to the length l of $\mathfrak{B}$.*

LEMMA 2.2. *If $\mathfrak{a}$ is an ideal of A which contains $\mathfrak{q}$, then $0 : (0 : \mathfrak{a}) = \mathfrak{a}$; i.e., the annihilating ideal of the module $0 : \mathfrak{a}$ is $\mathfrak{a}$. In the same way, if $\mathfrak{W}$ is a submodule of $\mathfrak{B}$, then $0 : (0 : \mathfrak{W}) = \mathfrak{W}$.*

It follows from lemma 2.2 that a submodule $\mathfrak{W}$ of $\mathfrak{B}$ is uniquely determined by its annihilating ideal $\mathfrak{a} = 0 : \mathfrak{W}$, since $\mathfrak{W} = 0 : \mathfrak{a}$. Consequently, different submodules of $\mathfrak{B}$ have different annihilating ideals. Furthermore, if $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two submodules of $\mathfrak{B}$ with the annihilating ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively, then the annihilating ideal of $\mathfrak{W}_3 = \mathfrak{W}_1 \cap \mathfrak{W}_2$ is equal to the sum $(\mathfrak{a}_1, \mathfrak{a}_2)$ of $\mathfrak{a}_1$ and $\mathfrak{a}_2$. In the first place, if $\mathfrak{a}_3$ is the annihilating ideal of $\mathfrak{W}_3$, then $\mathfrak{a}_3 = 0 : \mathfrak{W}_3$, while $0 : (\mathfrak{a}_1, \mathfrak{a}_2) = (0 : \mathfrak{a}_1) \cap (0 : \mathfrak{a}_2) = \mathfrak{W}_1 \cap \mathfrak{W}_2 = \mathfrak{W}_3$, according to lemma 2.2. Hence, $(\mathfrak{a}_1, \mathfrak{a}_2) = 0 : (0 : (\mathfrak{a}_1, \mathfrak{a}_2)) = 0 : \mathfrak{W}_3 = \mathfrak{a}_3$, which proves the following lemma.

LEMMA 2.3. *A submodule $\mathfrak{W}$ of $\mathfrak{B}$ is uniquely determined by its annihilating ideal. If $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two submodules of $\mathfrak{B}$ with respectively the annihilating ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$, then the annihilating ideal of $\mathfrak{W}_1 \cap \mathfrak{W}_2$ is $(\mathfrak{a}_1, \mathfrak{a}_2)$.*

Let $\mathfrak{a}$ be a divisor of $\mathfrak{q}$. Then there exists a composition series from $\mathfrak{q}$ to A which passes through $\mathfrak{a}$, say $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{a} = \mathfrak{q}_h \subset \mathfrak{q}_{h+1} \subset \ldots \subset \mathfrak{q}_l = A$. Hence, the length of the difference module $A - \mathfrak{a}$ is $l - h$. Since, according to lemma 2.1, $0 \subset 0 : \mathfrak{q}_{l-1} \subset \ldots \subset 0 : \mathfrak{a} \subset \ldots \subset 0 : \mathfrak{q}_1 \subset \mathfrak{B}$ is a composition series of $\mathfrak{B}$, the length of the module $0 : \mathfrak{a}$ is $l - h$. This proves lemma 2.4, which will be used in sec. 5 and which is the analogue of Theorem 6 of [3].

LEMMA 2.4. *If $\mathfrak{a}$ is a divisor of $\mathfrak{q}$, the length of the difference module $A - \mathfrak{a}$ is equal to the length of the submodule $0 : \mathfrak{a}$ of $\mathfrak{B}$. In the same way, if $\mathfrak{W}$ is a submodule of $\mathfrak{B}$, the length of $\mathfrak{W}$ is equal to the length of the difference module $A - (0 : \mathfrak{W})$. ($0 : \mathfrak{W}$ is the annihilating ideal of $\mathfrak{W}$.)*

REMARK 2.1. It is obvious that a submodule $\mathfrak{W}$ of a completely indecomposable module is itself completely indecomposable. Since the annihilating ideal of $\mathfrak{W}$ is $0 : \mathfrak{W}$, it follows from lemma 2.1 that the length of $\mathfrak{W}$ is equal to the length of the difference module $A - (0 : \mathfrak{W})$, which gives another proof of lemma 2.4.

3. The endomorphism ring of a completely indecomposable module. The purpose of this section is to prove the following theorem.

THEOREM 3.1. *The operator-endomorphism ring E of a completely indecomposable module $\mathfrak{B}$ with A as operator domain consists of the multiplications of the elements of $\mathfrak{B}$ with the elements of A. Hence, if $\mathfrak{q}$ is the annihilating ideal of $\mathfrak{B}$, then E is ring-isomorphic with the factor ring $A/\mathfrak{q}$.*

Proof. We shall consider E not as a ring, but as a module with A as right-operator domain, according to the following definitions.

$$V(M_1 + M_2) = VM_1 + VM_2, \text{ where } V \in \mathfrak{B} \text{ and } M_1, M_2 \in E;$$
$$V(M\mathfrak{a}) = (VM)\mathfrak{a}, \text{ where } V \in \mathfrak{B}, M \in E \text{ and } \mathfrak{a} \in A.$$

To any $\mathfrak{a} \in A$, there corresponds the endomorphism $M(\mathfrak{a})$, defined by $VM(\mathfrak{a}) = V\mathfrak{a}$ for all $V \in \mathfrak{B}$. (These endomorphisms $M(\mathfrak{a})$ are the multiplications mentioned in Theorem 3.1. Endomorphism and homomorphism will always mean operator-endomorphism and operator-homomorphism.) The correspondence $\mathfrak{a} \to M(\mathfrak{a})$ is clearly an operator-homomorphism from A into E, where both A and E are considered as A-modules. Since the kernel of this homomorphism is $\mathfrak{q}$, E contains a submodule E' which is operator isomorphic with the difference module $A - \mathfrak{q}$. All we have to show is that $E' = E$. However, E' is an A-module of finite length; namely, the length of E' is equal to the length of $A - \mathfrak{q}$ and hence, according to lemma 2.1, is equal to the length l of $\mathfrak{B}$. Consequently, all we have to prove is that the A-module E has a composition series of length l. Hereto, let $0 = \mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \ldots \subset \mathfrak{B}_{l-1} \subset \mathfrak{B}_l = \mathfrak{B}$ be a composition series of $\mathfrak{B}$. Let E_i, for $i = 0, \ldots, l$, be the submodule of E which consists of those endomorphisms which annihilate $\mathfrak{B}_i$; i.e., $VM = 0$ for all $V \in \mathfrak{B}_i$ and $M \in E_i$. Then, obviously $E_0 = E$ and E_l is the zero-endomorphism Ω, while $E = E_0 \supseteq E_1 \supseteq \ldots \supseteq E_{l-1} \supseteq E_l = \Omega$. We will prove that this series is a composition

series of E. It will be enough to show that, for any fixed integer i, where $1 \leq i \leq l$, the difference module $E_{i-1} - E_i$ is operator-isomorphic with the difference module $A - \mathfrak{p}$, where $\mathfrak{p}$ is the radical of 0 ($\mathfrak{p}$ is a maximal ideal of A according to sec. 1). Let $M \in E_{i-1}$, then $\mathfrak{B}_i M$ is a submodule of $\mathfrak{B}$ which is homomorphic with $\mathfrak{B}_i$. Since $\mathfrak{B}_{i-1} M = 0$, the length of $\mathfrak{B}_i M$ is at most 1. Consequently, since $\mathfrak{B}_1$ is the only submodule of $\mathfrak{B}$ of length 1, $\mathfrak{B}_i M \subseteq \mathfrak{B}_1$. Now, let V_1 be a fixed non-zero element of $\mathfrak{B}_1$ and let V_i be a fixed element of $\mathfrak{B}_i$ where V_i not $\in \mathfrak{B}_{i-1}$. Then, $\mathfrak{B}_1$ is a cyclic module with V_1 as generator and $\mathfrak{B}_i = (\mathfrak{B}_{i-1}, V_i)$. (This means that $\mathfrak{B}_i$ is generated by $\mathfrak{B}_{i-1}$ and V_i; i.e., an element of $\mathfrak{B}_i$ can always be written as $W_{i-1} + V_i \mathfrak{a}$ where $W_{i-1} \in \mathfrak{B}_{i-1}$ and $\mathfrak{a} \in A$.) It follows that there exists an $\mathfrak{a} \in A$ such that $V_i M = V_1 \mathfrak{a}$. Let $\bar{\mathfrak{a}}$ denote the coset of $\mathfrak{a}$ modulo $\mathfrak{p}$. Then, since $\mathfrak{p}$ is the annihilating ideal of V_1, the correspondence $M \to \bar{\mathfrak{a}}$ is an operator-homomorphism H from E_{i-1} into $A - \mathfrak{p}$. The kernel of H is clearly E_i and hence the difference module $E_{i-1} - E_i$ is operator-isomorphic with a submodule of $A - \mathfrak{p}$. Since the only submodules of $A - \mathfrak{p}$ are the zero-module and $A - \mathfrak{p}$ itself, all that remains to be shown is that $E_{i-1} - E_i$ is not the zero-module, i.e., that $E_{i-1} \supset E_i$. However, it follows from lemma 2.1 that $0 : \mathfrak{B}_i \subset 0 : \mathfrak{B}_{i-1}$ and consequently, there exists an element $\mathfrak{a}$ of A such that $\mathfrak{a} \in 0 : \mathfrak{B}_{i-1}$ and $\mathfrak{a}$ not $\in 0 : \mathfrak{B}_i$. The endomorphism $VM(\mathfrak{a}) = V\mathfrak{a}$, for all $V \in \mathfrak{B}$, is clearly an element of E_{i-1} which is not contained in E_i; hence Theorem 3.1 is proved.

REMARK 3.1. Any cyclic module $\mathfrak{B}$ which has a ring A as operator domain and which satisfies property I-1 is operator-isomorphic with the difference module $A - \mathfrak{a}$, where $\mathfrak{a}$ is the annihilating ideal of $\mathfrak{B}$. As is well known, the operator-endomorphisms of $A - \mathfrak{a}$ are just the multiplications of the elements of $A - \mathfrak{a}$ with the elements of A (see, for instance, [4], sec. 120) and hence the operator-endomorphism ring of $A - \mathfrak{a}$ is ring-isomorphic with the factor ring $A/\mathfrak{a}$. In other words, Theorem 3.1 always holds for a cyclic module. As is shown by example 7.3, a completely indecomposable module is not necessarily cyclic.

It follows immediately from Theorem 3.1 that if $\mathfrak{B}$ is a completely indecomsable module with A as operator domain, 0 as zero-module and $\mathfrak{p}$ as radical of 0, then the automorphism-group of $\mathfrak{B}$ consists of the multiplications of elements of $\mathfrak{B}$ with elements $\mathfrak{a} \in A$, where $\mathfrak{a}$ not $\in \mathfrak{p}$. The elements of $\mathfrak{p}$ give rise to the endomorphisms of $\mathfrak{B}$ with non-zero kernel.

COROLLARY 3.1. *Let $\mathfrak{B}$ be a completely indecomposable module and let $\mathfrak{W}$ be a submodule of $\mathfrak{B}$. Then, every operator-endomorphism M^* of $\mathfrak{W}$ can be extended to an operator-endomorphism M of $\mathfrak{B}$. If M^* is an automorphism of $\mathfrak{W} \neq 0$, M is an automorphism of $\mathfrak{B}$.*

Proof. Let A be the operator domain of $\mathfrak{B}$. Then $\mathfrak{W}$, as a submodule of a completely indecomposable module, is itself completely indecomposable and has A as operator domain. Hence, it follows from Theorem 3.1 that if M^* is an endomorphism of $\mathfrak{W}$, there exists an $\mathfrak{a} \in A$, such that $VM^* = V\mathfrak{a}$ for all

$V \in \mathfrak{W}$. The endomorphism $VM = V\alpha$ for all $V \in \mathfrak{V}$ is clearly an extension of M^* to an endomorphism M of $\mathfrak{V}$. If M^* is an automorphism of $\mathfrak{W} \neq 0$, α not $\in \mathfrak{p}^*$, where $\mathfrak{p}^*$ is the radical of the zero-module 0 of $\mathfrak{W}$. Hence, $\mathfrak{p}^*$ is the radical of the annihilating ideal $\mathfrak{q}^*$ of $\mathfrak{W}$ (see sec. 1). If $\mathfrak{q}$ is the annihilating ideal of $\mathfrak{V}$ and $\mathfrak{p}$ is the radical of $\mathfrak{q}$, then $\mathfrak{q} \subset \mathfrak{q}^* \subset A$ and hence $\mathfrak{p} \subseteq \mathfrak{p}^* \subset A$. Since $\mathfrak{p}$ and $\mathfrak{p}^*$ are maximal prime ideals of A, $\mathfrak{p} = \mathfrak{p}^*$ and hence α not $\in \mathfrak{p}$, which implies that M is an automorphism of $\mathfrak{V}$.

Corollary 3.1 can be used to characterize completely indecomposable modules. Precisely, *an indecomposable module $\mathfrak{V}$ with a ring A as operator domain, where conditions* I-1, I-2, *and* I-3 *are satisfied, is completely indecomposable if and only if every operator-endomorphism of any submodule $\mathfrak{W}$ of $\mathfrak{V}$ can be extended to an operator-endomorphism of $\mathfrak{V}$.* The "only if part" is proved by corollary 3.1. Now suppose that $\mathfrak{V}$ is indecomposable, that conditions I-1, I-2, I-3 are satisfied and that, if $\mathfrak{W}$ is any submodule of $\mathfrak{V}$, every operator-endomorphism of $\mathfrak{W}$ can be extended to an operator-endomorphism of $\mathfrak{V}$. Then, if $\mathfrak{W}$ is decomposable, say $\mathfrak{W} = \mathfrak{W}_1 \dot{+} \mathfrak{W}_2$, the projection P^* of $\mathfrak{W}$ on $\mathfrak{W}_1$ is an operator-endomorphism of $\mathfrak{W}$. (If $W \in \mathfrak{W}$, then $W = W_1 + W_2$, and $WP^* = W_1$.) This P^* could not be extended to an operator-endomorphism P of $\mathfrak{V}$, since P would then be a not-nilpotent endomorphism of $\mathfrak{V}$ with non-zero kernel which, according to Fitting's lemma (see [5], p. 11) is not possible since $\mathfrak{V}$ is indecomposable. Hence every submodule of $\mathfrak{V}$ is indecomposable, which means that $\mathfrak{V}$ is completely indecomposable.

Corollary 3.1 and the following lemma on arbitrary operator groups will be used in the proof of Theorem 5.1.

LEMMA 3.1. *Let $\mathfrak{V}$ and $\mathfrak{W}$ be two additive abelian groups with the same ring A as operator domain. (A is not necessarily commutative and none of the properties* I-1 *through* I-4 *are assumed.) Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two submodules of $\mathfrak{V}$ and let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be two submodules of $\mathfrak{W}$. Let I_1 be an operator-isomorphism between $\mathfrak{V}_1$ and $\mathfrak{W}_1$ and let I_2 be an operator-isomorphism between $\mathfrak{V}_2$ and $\mathfrak{W}_2$. ($\mathfrak{V}_1$, $\mathfrak{V}_2$, $\mathfrak{W}_1$ and $\mathfrak{W}_2$ all have A as operator domain.) Let I_1 map $\mathfrak{V}_3 = \mathfrak{V}_1 \cap \mathfrak{V}_2$ isomorphically onto $\mathfrak{W}_3 = \mathfrak{W}_1 \cap \mathfrak{W}_2$; i.e., I_1 induces an operator-isomorphism I^*_1 between $\mathfrak{V}_3$ and $\mathfrak{W}_3$. In the same way, let I_2 map $\mathfrak{V}_3$ isomorphically onto $\mathfrak{W}_3$; i.e., I_2 induces an operator-isomorphism I^*_2 between $\mathfrak{V}_3$ and $\mathfrak{W}_3$. Finally, let $I^*_1 = I^*_2 = I_3$. Then, there exists an operator-isomorphism I between $(\mathfrak{V}_1, \mathfrak{V}_2)$ and $(\mathfrak{W}_1, \mathfrak{W}_2)$ which is simultaneously an extension of I_1, I_2 and I_3.*

Proof. Let $V \in (\mathfrak{V}_1, \mathfrak{V}_2)$; hence $V = V_1 + V_2$ where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$. ($(\mathfrak{V}_1, \mathfrak{V}_2)$ denotes the submodule generated by $\mathfrak{V}_1$ and $\mathfrak{V}_2$.) We define $VI = V_1I_1 + V_2I_2$ and assert that I is an operator-isomorphism between $(\mathfrak{V}_1, \mathfrak{V}_2)$ and $(\mathfrak{W}_1, \mathfrak{W}_2)$. In the first place, we show that VI does not depend on the choice of V_1 and V_2. Let $V = V'_1 + V'_2$ where $V'_1 \in \mathfrak{V}_1$ and $V'_2 \in \mathfrak{V}_2$; then $V_1 - V'_1 = V'_2 - V_2$ is an element of $\mathfrak{V}_3$ and hence $(V_1 - V'_1)I_1 = (V'_2 - V_2)I_2$, which implies that $V_1I_1 - V'_1I_1 = V'_2I_2 - V_2I_2$ and hence that $VI = V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$. In the second place, $VI \in (\mathfrak{W}_1, \mathfrak{W}_2)$ since $V_1I_1 \in \mathfrak{W}_1$ and $V_2I_2 \in$

$\mathfrak{W}_2$. One can prove immediately that $(V + V')I = VI + V'I$ and $(V\mathfrak{a})I = (VI)\mathfrak{a}$, where V and V' are elements of $(\mathfrak{V}_1, \mathfrak{V}_2)$ and where $\mathfrak{a} \in A$, and also that I maps $(\mathfrak{V}_1, \mathfrak{V}_2)$ *onto* $(\mathfrak{W}_1, \mathfrak{W}_2)$. We now show that the kernel of I is the zero-element 0 of $\mathfrak{V}$. Hereto, let $VI = 0'$, where $V \in (\mathfrak{V}_1, \mathfrak{V}_2)$ and where $0'$ is the zero element of $\mathfrak{W}$. Then $V = V_1 + V_2$, where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, and $VI = V_1I_1 + V_2I_2 = 0'$ and hence $V_1I_1 = -V_2I_2$. Since $V_1I_1 \in \mathfrak{W}_1$ and $-V_2I_2 \in \mathfrak{W}_2$, V_1I_1 and $-V_2I_2$ are both elements of $\mathfrak{W}_3$. Since I_1 maps $\mathfrak{V}_1$ isomorphically onto $\mathfrak{W}_1$ and, at the same time, $\mathfrak{V}_2$ isomorphically onto $\mathfrak{W}_3$, $V_1 \in \mathfrak{V}_2$ and, in the same way, $-V_2 \in \mathfrak{V}_2$. Consequently, $V_1I_1 = V_1I_3$ and $-V_2I_2 = -V_2I_3$ and hence $V_1I_1 + V_2I_2 = V_1I_3 + V_2I_3 = (V_1 + V_2)I_3 = 0'$ which implies, since I_3 is an isomorphism, that $0 = V_1 + V_2 = V$. Finally, I is an extension of I_1, since if $V \in \mathfrak{V}_1$, then $VI = VI_1 + 0I_2 = VI_1$ and, in the same way, I is an extension of I_2 and I_3.

Let $\mathfrak{V}$, $\mathfrak{V}_1$, $\mathfrak{V}_2$, $\mathfrak{V}_3$, $\mathfrak{W}$, $\mathfrak{W}_1$, $\mathfrak{W}_2$, $\mathfrak{W}_3$ have the same meaning as in lemma 3.1. Another way of formulating lemma 3.1 is as follows.

If there exist an operator-isomorphism I_3 between $\mathfrak{V}_3$ and $\mathfrak{W}_3$, an operator-isomorphism I_1 between $\mathfrak{V}_1$ and $\mathfrak{W}_1$, and an operator-isomorphism I_2 between $\mathfrak{V}_2$ and $\mathfrak{W}_2$, where I_1 and I_2 are extensions of I_3, then I_1, I_2 and I_3 can be extended simultaneously to an operator-isomorphism I between $(\mathfrak{V}_1, \mathfrak{V}_2)$ and $(\mathfrak{W}_1, \mathfrak{W}_2)$.

4. Two lemmas of ring-representation theory. The following lemmas 4.1 and 4.2, which will be used in secs. **5** and **6**, are immediate corollaries of two lemmas proved by Professor R. Brauer in a course on group theory, given during the summer of 1947 at the University of Michigan. The proofs of lemmas 4.1 and 4.2, which are added here for the sake of completeness, can be derived immediately from Professor Brauer's proofs of the more general lemmas.

LEMMA 4.1. *Let $\mathfrak{V}$ be an additive abelian group with a ring A as operator domain where property* I-1 *is satisfied and where $\mathfrak{V}$ has a unique maximal submodule $\mathfrak{W}$. Let $\mathfrak{a}$ be the annihilating ideal of $\mathfrak{V}$. Then, $\mathfrak{V}$ is operator isomorphic with $A - \mathfrak{a}$.*

Proof. Since $\mathfrak{W}$ is a unique maximal submodule of $\mathfrak{V}$, $\mathfrak{W} \subset \mathfrak{V}$ and $\mathfrak{W}$ contains every proper submodule of $\mathfrak{V}$. Let $V \in \mathfrak{V}$, where V not $\in \mathfrak{W}$. The submodule VA of $\mathfrak{V}$ is not contained in $\mathfrak{W}$ since $V.1 = V$ not $\in \mathfrak{W}$. Hence, $VA = \mathfrak{V}$ which implies that $\mathfrak{V}$ is cyclic and hence operator isomorphic with $A - \mathfrak{a}$, where $\mathfrak{a}$ is the annihilating ideal of $\mathfrak{V}$.

LEMMA 4.2. *Let $\mathfrak{V}$ be an additive abelian group with a ring A as operator domain where property* I-1 *is satisfied. Let $\mathfrak{a}$ be the annihilating ideal of $\mathfrak{V}$ and suppose that $\mathfrak{V}$ has a submodule $\mathfrak{W}$ such that $\mathfrak{V} - \mathfrak{W}$ is operator isomorphic with $A - \mathfrak{a}$. Then, $\mathfrak{V}$ is decomposable in a direct sum, $\mathfrak{V} = \mathfrak{W} \dot{+} \mathfrak{V}_1$ where $\mathfrak{V}_1$ is operator-isomorphic with $A - \mathfrak{a}$.*

Proof. Let T denote the natural homomorphism from $\mathfrak{V}$ onto $\mathfrak{V} - \mathfrak{W}$; i.e., for any $V \in \mathfrak{V}$, VT is the coset of V modulo $\mathfrak{W}$. Let Q denote the isomorphism

between $\mathfrak{B} - \mathfrak{W}$ and $A - \mathfrak{a}$, whose existence is stated in lemma 4.2. Then $P = TQ$ (first T, then Q) is an operator-homomorphism from $\mathfrak{B}$ *onto* $A - \mathfrak{a}$ with kernel $\mathfrak{W}$. Let V_1 be an element of $\mathfrak{B}$ such that $V_1P = \bar{1}$, where $\bar{1}$ is the coset of the unit element 1 of A modulo $\mathfrak{a}$. We claim that we can take V_1A for the $\mathfrak{B}_1$ of lemma 4.2. (V_1A is the submodule of $\mathfrak{B}$ which consists of the elements $V_1\mathfrak{a}$, where $\mathfrak{a} \in A$.) In the first place, P maps V_1A isomorphically onto $A - \mathfrak{a}$, since $(V_1\mathfrak{a})P = (V_1P)\mathfrak{a} = \bar{1}\mathfrak{a} = \bar{\mathfrak{a}}$ for any $\mathfrak{a} \in A$ (bars denote cosets modulo $\mathfrak{a}$). This proves already that V_1A is operator-isomorphic with $A - \mathfrak{a}$ and that $\mathfrak{W} \cap V_1A = 0$, where 0 is the zero-element of $\mathfrak{B}$. In the second place, for any $V \in \mathfrak{B}$, we can find an $\mathfrak{a} \in A$ such that $VP = (V_1\mathfrak{a})P$ and hence such that $V - V_1\mathfrak{a} = W$, where W lies in the kernel $\mathfrak{W}$ of P. Consequently, $V = W + V_1\mathfrak{a}$ which proves that $\mathfrak{B} = \mathfrak{W} \dotplus \mathfrak{B}_1$.

5. Criterion of isomorphism of completely indecomposable modules. We can now prove the following criterion that two completely indecomposable modules be isomorphic.

THEOREM 5.1. *Two completely indecomposable modules with the same operator domain are operator-isomorphic if and only if they have the same annihilating ideals.*

Proof. Let $\mathfrak{B}$ and $\mathfrak{W}$ be two completely indecomposable modules with the same operator domain A. Hence, properties I-1, I-2, I-3 and I-4 are satisfied. It is obvious that if $\mathfrak{B}$ and $\mathfrak{W}$ are operator-isomorphic, they have the same annihilating ideal. Hence we assume that $\mathfrak{B}$ and $\mathfrak{W}$ have the same annihilating ideal $\mathfrak{q}$ and prove that then $\mathfrak{B}$ and $\mathfrak{W}$ are operator-isomorphic.

Since the lengths of $\mathfrak{B}$ and of $\mathfrak{W}$ are both equal to the length of the difference module $A - \mathfrak{q}$ (see lemma 2.1), let l be the common length of $\mathfrak{B}$, $\mathfrak{W}$ and $A - \mathfrak{q}$. If $l = 1$, both $\mathfrak{B}$ and $\mathfrak{W}$ are cyclic and hence operator-isomorphic with $A - \mathfrak{q}$ and hence isomorphic with each other. Consequently, we assume that $l > 1$ and we make the induction hypothesis that Theorem 5.1 has been proved for $l = 1, 2, \ldots, l - 1$. There are two cases to be considered.

Case 1. *One of the two modules, say* $\mathfrak{B}$, *has only one maximal submodule.* Since $\mathfrak{B}$ has a unique maximal submodule, it follows from lemma 4.1 that $\mathfrak{B}$ is operator-isomorphic with $A - \mathfrak{q}$. This implies in particular that $A - \mathfrak{q}$ is completely indecomposable and hence that $A - \mathfrak{q}$ has a unique minimal submodule. Consequently, there exists only one ideal $\mathfrak{q}_1$ such that $\mathfrak{q} \subset \mathfrak{q}_1$ and where the length of the difference module $A - \mathfrak{q}_1$ is $l - 1$. If $0'$ is the zero-module of $\mathfrak{W}$, the length of $\mathfrak{W}_{l-1} = 0' : \mathfrak{q}_1$ is $l - 1$ (see lemma 2.4) and hence $\mathfrak{W}_{l-1}$ is a maximal submodule of $\mathfrak{W}$. We assert that $\mathfrak{W}_{l-1}$ is the only maximal submodule of $\mathfrak{W}$. Let $\mathfrak{W}'_{l-1}$ be any maximal submodule of $\mathfrak{W}$ where $\mathfrak{q}'_1$ is the annihilating ideal of $\mathfrak{W}'_{l-1}$. Then the length of $\mathfrak{W}'_{l-1}$ is $l - 1$ and hence the length of $A - \mathfrak{q}'_1$ is also $l - 1$ (see lemma 2.4); consequently, $\mathfrak{q}'_1 = \mathfrak{q}_1$ and, according to lemma 2.2, $\mathfrak{W}'_{l-1} = 0' : \mathfrak{q}'_1 = 0' : \mathfrak{q}_1 = \mathfrak{W}_{l-1}$, which proves the assertion. Lemma 4.1 then implies that $\mathfrak{W}$ is also operator-isomorphic with $A - \mathfrak{q}$ and hence that $\mathfrak{B}$ and $\mathfrak{W}$ are isomorphic.

Case 2. Both modules $\mathfrak{B}$ and $\mathfrak{W}$ have at least two different maximal submodules. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two different maximal submodules of $\mathfrak{B}$, which implies that $\mathfrak{B} = (\mathfrak{B}_1, \mathfrak{B}_2)$. The length of both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is $l - 1$. If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the annihilating ideals of respectively $\mathfrak{B}_1$ and $\mathfrak{B}_2$, then $\mathfrak{q}_1 = 0 : \mathfrak{B}_1$ and $\mathfrak{q}_2 = 0 : \mathfrak{B}_2$, where 0 is the zero-module of $\mathfrak{B}$. Furthermore, $\mathfrak{q}_1 \neq \mathfrak{q}_2$ according to lemma 2.3, and the length of both the difference modules $A - \mathfrak{q}_1$ and $A - \mathfrak{q}_2$ is $l - 1$ (see lemma 2.4). Consider the submodules $\mathfrak{W}_1 = 0' : \mathfrak{q}_1$ and $\mathfrak{W}_2 = 0' : \mathfrak{q}_2$ of $\mathfrak{W}$, where $0'$ is the zero-module of $\mathfrak{W}$. According to lemma 2.3, $\mathfrak{W}_1 \neq \mathfrak{W}_2$, and according to lemma 2.4 the length of both $\mathfrak{W}_1$ and $\mathfrak{W}_2$ is $l - 1$; i.e., $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two distinct, maximal submodules of $\mathfrak{W}$, which implies that $\mathfrak{W} = (\mathfrak{W}_1, \mathfrak{W}_2)$. Since $\mathfrak{B}_1$ and $\mathfrak{W}_1$ are completely indecomposable modules (a submodule of a completely indecomposable module is obviously completely indecomposable) and have the same annihilating ideal $\mathfrak{q}_1$, it follows from the induction hypothesis that $\mathfrak{B}_1$ and $\mathfrak{W}_1$ are operator-isomorphic. In the same way, $\mathfrak{B}_2$ and $\mathfrak{W}_2$ are operator-isomorphic. Hence, let I_1 be an operator-isomorphism which maps $\mathfrak{B}_1$ onto $\mathfrak{W}_1$ and let I_2 be an operator-isomorphism which maps $\mathfrak{B}_2$ onto $\mathfrak{W}_2$. We claim that I_1 maps $\mathfrak{B}_3 = \mathfrak{B}_1 \cap \mathfrak{B}_2$ isomorphically onto $\mathfrak{W}_3 = \mathfrak{W}_1 \cap \mathfrak{W}_2$ and that I_2 maps $\mathfrak{B}_3$ isomorphically onto $\mathfrak{W}_3$. In the first place, the annihilating ideal of $\mathfrak{B}_3$ and $\mathfrak{W}_3$ is $(\mathfrak{q}_1, \mathfrak{q}_2)$ according to lemma 2.3. In the second place, both $\mathfrak{B}_3 I_1$ and $\mathfrak{B}_3 I_2$ are submodules of $\mathfrak{W}$ which are operator-isomorphic with $\mathfrak{B}_3$. Hence these modules have $(\mathfrak{q}_1, \mathfrak{q}_2)$ as annihilating ideal, which implies, according to lemma 2.3, that $\mathfrak{B}_3 I_1 = \mathfrak{B}_3 I_2 = \mathfrak{W}_3$. Let I^*_1 and I^*_2 be the operator-isomorphisms which map $\mathfrak{B}_3$ onto $\mathfrak{W}_3$ and which are induced respectively by I_1 and I_2; i.e., if $V \in \mathfrak{B}_3$, then $VI^*_1 = VI_1 \in \mathfrak{W}_3$ and $VI^*_2 = VI_2 \in \mathfrak{W}_3$. We cannot as yet apply lemma 3.1 since it may be that $I^*_1 \neq I^*_2$. However, we shall change I_1 into a new operator-isomorphism J_1 such that $J^*_1 = I^*_2$. Hereto, let $(I^*_1)^{-1}$ be the inverse of I^*_1; hence, $(I^*_1)^{-1}$ maps $\mathfrak{W}_3$ operator-isomorphically onto $\mathfrak{B}_3$. Then, $(I^*_1)^{-1} I^*_2$ is clearly an operator-automorphism, say H^*, of $\mathfrak{W}_3$ onto itself. Since $\mathfrak{W}_3 \subseteq \mathfrak{W}_1$, it follows from corollary 3.1 that there exists an operator-automorphism H of $\mathfrak{W}_1$ onto itself, where H is an extension of H^*; i.e., if $W \in \mathfrak{W}_3$, then $WH = WH^*$. For J_1 we then take the operator-isomorphism $J_1 = I_1 H$ which clearly maps $\mathfrak{B}_1$ onto $\mathfrak{W}_1$. Furthermore, if J^*_1 is the operator-isomorphism between $\mathfrak{B}_3$ and $\mathfrak{W}_3$, induced by J_1, then $J^*_1 = I^*_2$ because, if $V \in \mathfrak{B}_3$, then $VJ^*_1 = VI_1 H = VI^*_1 H$ and, since $VI^*_1 \in \mathfrak{W}_3$, $VI^*_1 H = VI^*_1 H^* = VI^*_1 (I^*_1)^{-1} I^*_2 = VI^*_2 = VI_2$. We can then conclude from lemma 3.1 that there exists an operator-isomorphism between $(\mathfrak{B}_1, \mathfrak{B}_2) = \mathfrak{B}$ and $(\mathfrak{W}_1, \mathfrak{W}_2) = \mathfrak{W}$ which proves Theorem 5.1.

REMARK 5.1. The module $\mathfrak{B}$ in Theorem 5.1 can be considered as a representation space of the representation $\alpha \to M(\alpha)$ of A, where $\alpha \in A$ and where $M(\alpha)$ is the operator-endomorphism $VM(\alpha) = V\alpha$ of $\mathfrak{B}$. This representation of A is faithful if and only if the annihilating ideal $\mathfrak{q}$ of $\mathfrak{B}$ is the zero-ideal. Furthermore, according to Theorem 5.1, two completely indecomposable modules are operator-isomorphic if they have the same operator domain A and each has the zero-ideal as annihilating ideal. Consequently, as stated in the intro-

duction, two faithful representations of A, both of which give rise to completely indecomposable representation spaces, are equivalent.

The following corollary of Theorem 5.1 will be used in sec. 6.

COROLLARY 5.1. *A completely indecomposable module is cyclic if and only if its annihilating ideal is intersection-irreducible.*

Proof. Let $\mathfrak{B}$ be a completely indecomposable module with A as operator domain and $\mathfrak{q}$ as annihilating ideal. If $\mathfrak{B}$ is cyclic, $\mathfrak{B}$ is operator-isomorphic with the difference module $A - \mathfrak{q}$, which implies that then $A - \mathfrak{q}$ is completely indecomposable and hence $\mathfrak{q}$ is intersection-irreducible (see remark 1.2). Conversely, if $\mathfrak{q}$ is intersection-irreducible, $A - \mathfrak{q}$ is completely indecomposable. Consequently, $\mathfrak{B}$ and $A - \mathfrak{q}$ are then two completely indecomposable modules with the same annihilating ideal $\mathfrak{q}$. Theorem 5.1 then implies that $\mathfrak{B}$ and $A - \mathfrak{q}$ are operator-isomorphic and hence that $\mathfrak{B}$ is then cyclic.

DEFINITION 5.1. *An additive abelian group $\mathfrak{B}$ with a ring A as operator domain is called a regular module if property* I-1 *is satisfied and if $\mathfrak{B}$ is the direct sum of a finite number of completely indecomposable submodules.*

REMARK 5.2. The term "regular" is taken from [3], where regular rings are defined. A regular ring, considered as a module with itself as operator domain, is a regular module in the sense of definition 5.1.

Let $\mathfrak{B}$ be any additive abelian group with a ring A as operator domain, where conditions I-1, I-2 and I-3 are satisfied. Then, $\mathfrak{B}$ is a direct sum of indecomposable submodules $\mathfrak{B} = \mathfrak{B}_1 \dotplus \ldots \dotplus \mathfrak{B}_h$, where $\mathfrak{B}_i$ is an indecomposable module for $i = 1, \ldots, h$. If $\mathfrak{B} = \mathfrak{B}'_1 \dotplus \ldots \dotplus \mathfrak{B}'_{h'}$ is another decomposition of $\mathfrak{B}$ into a direct sum of indecomposable submodules, $h = h'$ and, after a suitable ordering, $\mathfrak{B}_i$ is operator-isomorphic with $\mathfrak{B}'_i$ for $i = 1, \ldots, h$, according to the Krull-Schmidt theorem (see [5], p. 12). Consequently, the annihilating ideals of $\mathfrak{B}_1, \ldots, \mathfrak{B}_h$ are completely determined by $\mathfrak{B}$ and do not depend on the particular decomposition of $\mathfrak{B}$ which is chosen.

DEFINITION 5.2. *Let $\mathfrak{B}$ be an additive abelian group with a ring A as operator domain where conditions* I-1, I-2 *and* I-3 *are satisfied. Let $\mathfrak{B} = \mathfrak{B}_1 \dotplus \ldots \dotplus \mathfrak{B}_h$ be a decomposition of $\mathfrak{B}$ into a direct sum of indecomposable submodules. Then, the annihilating ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ respectively of $\mathfrak{B}_1, \ldots, \mathfrak{B}_h$ are called the elementary ideals of $\mathfrak{B}$.*

REMARK 5.3. Since $\mathfrak{B}_1, \ldots, \mathfrak{B}_h$ are indecomposable modules with composition sequences of finite length, the elementary ideals are always primary ideals of A with maximal associated primes (see sec. 1). The reason for calling $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ the elementary ideals of $\mathfrak{B}$, rather than the elementary divisors of $\mathfrak{B}$, is that $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ correspond to the primary factors of the classical elementary divisors and not to the elementary divisors themselves (see remark 6.1).

Since conditions I-1, I-2 and I-3 are clearly satisfied in a regular module, the elementary ideals of a regular module are defined by definition 5.2. The following theorem is an immediate corollary of Theorem 5.1 and constitutes an

extension of the classical theory of elementary divisors to regular modules (see sec. 6).

THEOREM 5.2. *Two regular modules with the same operator domain are operator-isomorphic if and only if they have the same elementary ideals.*

The following corollary is an immediate consequence of corollary 5.1.

COROLLARY 5.2. *A regular module $\mathfrak{B}$ with A as operator domain and whose elementary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ are intersection-irreducible ideals of A is a direct sum of cyclic submodules; namely, $\mathfrak{B}$ is operator-isomorphic with the direct sum of the h cyclic modules $A - \mathfrak{q}_1, \ldots, A - \mathfrak{q}_h$.*

Proof. $\mathfrak{B}$ is a direct sum $\mathfrak{B} = \mathfrak{B}_1 \dot{+} \ldots \dot{+} \mathfrak{B}_h$, where $\mathfrak{B}_i$ is a completely indecomposable submodule of $\mathfrak{B}$ with $\mathfrak{q}_i$ as annihilating ideal for $i = 1, \ldots, h$. Consequently, according to corollary 5.1, $\mathfrak{B}_i$ is cyclic and operator-isomorphic with $A - \mathfrak{q}_i$, which proves corollary 5.2.

6. The classical elementary divisors. We first derive the following lemma whose proof is modelled after Professor R. Brauer's lecture "The Normal Form of a Matrix," given during the summer of 1947 at the University of Michigan.

LEMMA 6.1. *Let $\mathfrak{B}$ be an additive abelian group with a ring A as operator domain where conditions* I-1, I-2 *and* I-3 *are satisfied. Furthermore, let $\mathfrak{B}$ be an indecomposable module and let the annihilating ideal $\mathfrak{q}$ of $\mathfrak{B}$ be an intersection-irreducible ideal of A. Then, $\mathfrak{B}$ is a cyclic module and is consequently operator-isomorphic with $A - \mathfrak{q}$, which implies that $\mathfrak{B}$ is completely indecomposable.*

Proof. Since $\mathfrak{q}$ is intersection-irreducible, the difference module $A - \mathfrak{q}$ is completely indecomposable (see remark 1.2); hence all we have to show is that $\mathfrak{B}$ is cyclic. Hereto we show first that, independent of whether $\mathfrak{B}$ is indecomposable or not, $\mathfrak{B}$ contains a submodule $\mathfrak{W}$ such that $\mathfrak{B} - \mathfrak{W}$ is operator-isomorphic with $A - \mathfrak{q}$. Let S be the collection of submodules of $\mathfrak{B}$ whose fundamental ideals are equal to $\mathfrak{q}$; i.e., $\mathfrak{W}' \in S$ if and only if $\mathfrak{W}' : \mathfrak{B} = \mathfrak{q}$. Since S contains the zero-module of $\mathfrak{B}$, S is not empty and hence I-2 assures that S contains a maximal element $\mathfrak{W}$. We assert that $\mathfrak{W}$ has the required property. We first show that $\mathfrak{W}$ is intersection-irreducible. Suppose that $\mathfrak{W} = \mathfrak{W}_1 \cap \mathfrak{W}_2$, where $\mathfrak{W} \subset \mathfrak{W}_1 \subset \mathfrak{B}$ and $\mathfrak{W} \subset \mathfrak{W}_2 \subset \mathfrak{B}$. Then, if $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the fundamental ideals respectively of $\mathfrak{W}_1$ and $\mathfrak{W}_2$, $\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2$ and, since $\mathfrak{W}$ is maximal in S, $\mathfrak{q} \subset \mathfrak{q}_1$ and $\mathfrak{q} \subset \mathfrak{q}_2$. This, however, is impossible since $\mathfrak{q}$ is intersection-irreducible, which shows that $\mathfrak{W}$ is intersection-irreducible. Hence the zero-module of the difference module $\mathfrak{B} - \mathfrak{W}$ is intersection-irreducible. Since conditions I-1, I-2 and I-3 are clearly satisfied by $\mathfrak{B} - \mathfrak{W}$, it follows from remark 1.2 that $\mathfrak{B} - \mathfrak{W}$ is a completely indecomposable module. Since the annihilating ideal of $\mathfrak{B} - \mathfrak{W}$ is $\mathfrak{W} : \mathfrak{B} = \mathfrak{q}$ and $\mathfrak{q}$ is intersection-irreducible, it follows from corollary 5.1 that $\mathfrak{B} - \mathfrak{W}$ is cyclic and consequently operator-isomorphic with $A - \mathfrak{q}$. We can now conclude from lemma 4.2 that $\mathfrak{B}$ is decomposable in a direct sum $\mathfrak{B} = \mathfrak{B}_1 \dot{+} \mathfrak{B}_2$, where $\mathfrak{B}_1$ is operator-isomorphic

with $A - \mathfrak{q}$. Hence, since $\mathfrak{B}$ is assumed to be indecomposable, $\mathfrak{B} = \mathfrak{B}_1$ and $\mathfrak{B}$ is operator-isomorphic with $A - \mathfrak{q}$ which shows that $\mathfrak{B}$ is cyclic.

The following corollary extends corollary 5.2 from regular modules to arbitrary modules which satisfy conditions I-1, I-2, I-3.

COROLLARY 6.1. *Let $\mathfrak{B}$ be an additive abelian group which has a ring A as operator domain and which satisfies conditions* I-1, I-2 *and* I-3. *Suppose that the elementary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ of $\mathfrak{B}$ are intersection-irreducible. Then $\mathfrak{B}$ is a direct sum of cyclic sub-modules. Namely, $\mathfrak{B}$ is operator-isomorphic with the direct sum of the h cyclic modules $A - \mathfrak{q}_1, \ldots, A - \mathfrak{q}_h$.*

Proof. $\mathfrak{B}$ is a direct sum $\mathfrak{B} = \mathfrak{B}_1 \dotplus \ldots \dotplus \mathfrak{B}_h$, where $\mathfrak{B}_i$ is an indecomposable module with $\mathfrak{q}_i$ as annihilating ideal for $i = 1, \ldots, h$. Consequently, according to lemma 6.1, $\mathfrak{B}_i$ is cyclic and operator-isomorphic with $A - \mathfrak{q}_i$, which proves corollary 6.1.

In the classical elementary divisor theory one deals with additive abelian groups which satisfy conditions I-1 and I-2 and have a ring A as operator domain, where A is either a Euclidean domain (see [4], sec. 108), a principal ideal ring, or the ring of integers of an algebraic number field (see [6]). All these rings have the property that every non-zero primary ideal, i.e., every primary ideal whose associated prime is maximal, is intersection-irreducible. Consequently, the following corollary, which follows immediately from corollary 6.1, contains that part of the classical elementary divisor theory which deals with modules in which the descending chain condition is satisfied; that is with modules whose annihilating ideal is not the zero-ideal.

COROLLARY 6.2. *Let $\mathfrak{B}$ be an additive abelian group with a ring A as operator domain where conditions* I-1, I-2 *and* I-3 *are satisfied. Suppose that every primary ideal of A whose associated prime ideal is maximal, is intersection-irreducible. Then $\mathfrak{B}$ is a direct sum of cyclic modules. Namely, $\mathfrak{B}$ is operator-isomorphic with the direct sum of the h cyclic modules $A - \mathfrak{q}_1, \ldots, A - \mathfrak{q}_h$, where $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ are the elementary ideals of $\mathfrak{B}$.*

REMARK 6.1. In the rings which occur in the classical elementary divisor theory, every primary ideal is a power of its associated prime ideal. Let A be such a ring. Then, if all the elementary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ of corollary 6.2 have the same associated prime ideal, $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \ldots \subseteq \mathfrak{q}_h$ after a suitable ordering. In other words, $\mathfrak{B}$ is then the direct sum of cyclic submodules whose annihilating ideals divide each other, and the elementary ideals are then exactly the classical elementary divisors of $\mathfrak{B}$. If among the associated prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ of $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ different ones occur, then $(\mathfrak{p}_i, \mathfrak{p}_j) = A$ whenever $\mathfrak{p}_i \neq \mathfrak{p}_j$. As a result one can easily show how the cyclic modules of corollary 6.2 can then be combined to give again the classical statement that $\mathfrak{B}$ is the direct sum of cyclic modules whose annihilating ideals $\mathfrak{a}_1 \subseteq \ldots \subseteq \mathfrak{a}_k$ divide each other. These $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ are then the classical elementary divisors and the $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ are the primary factors of the elementary divisors. This implies in particular that the $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ and the $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ determine each other completely, from

which it follows that two modules are operator-isomorphic if and only if they have the same elementary divisors.

REMARK 6.2. Let $\mathfrak{B}$ be a regular module with a ring A as operator domain and $\mathfrak{q}$ as annihilating ideal. If, as in the case of the classical elementary divisor theory, $\mathfrak{B}$ is the direct sum of cyclic submodules whose annihilating ideals divide each other, then the centre C of the operator-endomorphism ring of $\mathfrak{B}$ consists of the multiplications of the elements of $\mathfrak{B}$ with the elements of A, and consequently C is ring-isomorphic with $A/\mathfrak{q}$. (This is the actual content of [7]. If A is the ring which consists of the scalar polynomials of a square matrix M, this statement is equivalent with saying that the matrices which commute with all matrices which commute with M, are the scalar polynomials of M.) The same statement about the centre C of the operator-endomorphism ring of $\mathfrak{B}$ can obviously be made, according to Theorem 3.1, if $\mathfrak{B}$ is a completely indecomposable module, that is a regular module which is indecomposable. Whether the same statement is also correct for the centre of the operator-endomorphism ring of an arbitrary regular module is unsolved.

7. Examples of completely indecomposable and regular modules.

EXAMPLE 7.1. *The classical case.* Let A be a commutative principal ideal ring with unit element and without divisors of zero or let A be the ring of integers of an algebraic number field of finite degree. Let $\mathfrak{A}$ be the m-dimensional column vector space which consists of the columns of length m whose components are elements of A. Let $\mathfrak{B}$ be a submodule of $\mathfrak{A}$ of rank m. Then, the difference module $\mathfrak{B} = \mathfrak{A} - \mathfrak{B}$ is a regular module with A as operator domain. In the first place, it is well known that conditions I-1 and I-2 are satisfied in $\mathfrak{B}$. In the second place, the annihilating ideal of $\mathfrak{B}$ is not the zero-ideal since the rank of $\mathfrak{B}$ is m. This implies, since the non-zero associated prime ideals of A are maximal ideals, that condition I-3 is also satisfied in $\mathfrak{B}$. We can then conclude from corollary 6.2 that $\mathfrak{B}$ is regular. In this classical case, according to the same corollary, $\mathfrak{B}$ is indecomposable if and only if $\mathfrak{B}$ is cyclic and hence operator-isomorphic with $A - \mathfrak{q}$ where $\mathfrak{q}$ is the annihilator of $\mathfrak{B}$. Consequently, the completely indecomposable modules which occur in this case are all cyclic modules.

EXAMPLE 7.2. *Irreducible ideals.* Let A be a commutative ring with unit element whose ideals satisfy the ascending chain condition. Let $\mathfrak{q}$ be an intersection-irreducible ideal of A whose associated prime $\mathfrak{p}$ is a maximal ideal (an intersection-irreducible ideal is necessarily primary). Then conditions I-1, I-2 and I-3 are satisfied in the difference module $\mathfrak{B} = A - \mathfrak{q}$ and, according to remark 1.2, $\mathfrak{B}$ is a completely indecomposable module with A as operator domain. If $\mathfrak{p}$ is not maximal, a completely indecomposable module can be obtained by constructing the ring of quotients of the ring $A/\mathfrak{q}$, where the non-divisors of zero of $A/\mathfrak{q}$ are admitted as denominators (see [3], p. 215).

Regular modules can of course be constructed at will by forming direct sums

of the modules of examples 7.1 and 7.2. All the completely indecomposable modules mentioned in the previous examples are cyclic modules. The author is indebted to Professor R. M. Thrall for the following example which enables us to construct non-cyclic completely indecomposable modules at will.

EXAMPLE 7.3. *Duals of vector spaces.* Let $A = K[x_1, \ldots, x_n]$ be a polynomial domain in n variables $x_1, \ldots, x_n$ where K is a field. Let $\mathfrak{q}$ be a primary ideal of A whose associated prime ideal $\mathfrak{p}$ is maximal (i.e., $\mathfrak{p}$ is a zero-dimensional prime ideal of A). Then, the difference module $\mathfrak{A} = A - \mathfrak{q}$ is a (K, A)-module which, with respect to K, is a vector space of finite rank, say m. The dual vector space $\mathfrak{B}$ of $\mathfrak{A}$ consists, according to [8], p. 558, of the linear functionals from $\mathfrak{A}$ into K where, if $V \in \mathfrak{B}$ and $\alpha \in A$, the linear functional $V\alpha$ is defined by $(V\alpha)(x) = V(x\alpha)$ for all $x \in \mathfrak{A}$. It follows that $\mathfrak{B}$ is again a (K, A)-module which, with respect to K, has finite rank m, while the annihilating ideal of $\mathfrak{B}$ is $\mathfrak{q}$. Since $\mathfrak{B}$ is an A-module with finite K-rank, conditions I-1, I-2 and I-3 are satisfied in $\mathfrak{B}$. Furthermore, to every submodule $\mathfrak{B}$ of $\mathfrak{A}$, one can associate the submodule $\mathfrak{W}(\mathfrak{B})$ of $\mathfrak{B}$ which consists of the functionals which vanish on $\mathfrak{B}$. According to [8], this establishes a one to one correspondence between the submodules of $\mathfrak{A}$ and the submodules of $\mathfrak{B}$, such that $\mathfrak{B}$ consists of all the elements of $\mathfrak{A}$ which are annihilated by the functionals of $\mathfrak{W}(\mathfrak{B})$ and such that $\mathfrak{B}_1 \subset \mathfrak{B}_2$ implies $\mathfrak{W}(\mathfrak{B}_1) \supset \mathfrak{W}(\mathfrak{B}_2)$. It follows easily that, since the A-module $\mathfrak{A}$ has the unique maximal submodule $\mathfrak{B}_0 = \mathfrak{p} - \mathfrak{q}$, the A-module $\mathfrak{B}$ has the unique minimal submodule $\mathfrak{W}_0 = \mathfrak{W}(\mathfrak{B}_0)$ and hence that $\mathfrak{B}$ is a completely indecomposable A-module (see remark 1.1). Since $\mathfrak{B}$ has $\mathfrak{q}$ as annihilating ideal, $\mathfrak{B}$ cannot be cyclic if we choose for $\mathfrak{q}$ an ideal which is not intersection-irreducible, according to corollary 5.1. For instance, if $\mathfrak{q}$ is the ideal (x^2, xy, y^2) of the polynomial domain $A = K[x, y]$, then the dual vector space of the vector space $A - \mathfrak{q}$ is a non-cyclic, completely indecomposable, A-module.

REMARK 7.1. Let $\mathfrak{q}$ be a primary ideal with maximal associated prime of the polynomial ring $A = K[x_1, \ldots, x_n]$. The factor ring $A/\mathfrak{q}$ is then a "completely primary" commutative ring. (See [9], p. 96, for the notion of a completely primary ring.) At the same time, $A/\mathfrak{q}$ is an algebra with finite rank with respect to the field K. Conversely, every commutative completely primary ring which is at the same time an algebra with finite rank with respect to a field K, can be obtained in this way. Furthermore, we have seen in example 7.3, that the dual vector space $\mathfrak{B}$ of the vector space $A - \mathfrak{q}$ is a completely indecomposable A-module with $\mathfrak{q}$ as annihilating ideal. Hence, if $\mathfrak{B}$ is considered as a module with $A/\mathfrak{q}$ as operator domain, then $\mathfrak{B}$ is a faithful, completely indecomposable representation space of $A/\mathfrak{q}$. Consequently we see that *every commutative, completely primary ring which is, at the same time, an algebra of finite rank with respect to a field K has a faithful, completely indecomposable representation space.* Conversely, it follows easily from sec. 1 that, if a ring A has a faithful, completely indecomposable representation space, then A is a commutative, completely primary ring. It will be shown in sec. 9 that every commutative, completely primary ring A has a faithful, completely

indecomposable representation space, even if A is not an algebra of finite rank with respect to a field.

REMARK 7.2. If in condition I-1 we allow A to be a non-commutative ring, then the conditions I-1, I-2, I-3 and I-4 define the notion of a completely indecomposable module with a not necessarily commutative ring A as operator domain. If A is a (not necessarily commutative) completely primary ring, which is at the same time an algebra with finite rank m with respect to a field K, Theorem 5.1 still holds; that is, two faithful, completely indecomposable representation spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of A are A-isomorphic. This can be shown easily by means of the construction of dual vector spaces, as was pointed out by Professor Thrall. In the first place, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are easily seen to be vector spaces of finite rank m with respect to K; hence we can construct the dual vector spaces $\mathfrak{B}^*_1$ and $\mathfrak{B}^*_2$ of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have unique minimal submodules, $\mathfrak{B}^*_1$ and $\mathfrak{B}^*_2$ have unique maximal submodules; therefore $\mathfrak{B}^*_1$ and $\mathfrak{B}^*_2$ are both A-isomorphic with A, considered as an A-module (lemma 4.1 holds also for non-commutative rings). Hence, $\mathfrak{B}^*_1$ and $\mathfrak{B}^*_2$ are A-isomorphic, which implies that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are A-isomorphic. It is not known whether Theorem 5.1 remains valid for non-commutative operator domains which are not algebras of finite rank with respect to a field.

8. Interlacing of modules. This section is self-contained and describes a new composition of modules, called interlacing of modules, which will be used in the proof of Theorem 9.1. In order to stress the generality of this composition, which is a generalization of the direct sum of modules, we describe it also for non-commutative operator domains.

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two A-modules, that is any two additive abelian groups with the ring A as right operator domain. We make no assumptions whatsoever about chain conditions or about A; in particular, we do not assume that A is commutative or has a unit element. Let $\mathfrak{W}_1$ be a submodule of $\mathfrak{B}_1$ and $\mathfrak{W}_2$ a submodule of $\mathfrak{B}_2$, where $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are isomorphic; let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. (*Module, of course, always means A-module and isomorphism always means A-isomorphism.*) We want to construct a new module $\mathfrak{B}$ which, intuitively speaking, can be thought of as having been obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, where this interlacing is to be carried out by identifying the elements of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ which correspond under J. Algebraically, this means that we want $\mathfrak{B}$ to have the following three properties.

8.1 *There exists an isomorphism I_1 from $\mathfrak{B}_1$ onto a submodule $\mathfrak{B}'_1$ of $\mathfrak{B}$ and an isomorphism I_2 from $\mathfrak{B}_2$ onto a submodule $\mathfrak{B}'_2$ of $\mathfrak{B}$.*

8.2 $\mathfrak{B} = (\mathfrak{B}'_1, \mathfrak{B}'_2)$.

8.3 *If J_1 is the contraction of I_1 on $\mathfrak{W}_1$ and J_2 the contraction of I_2 on $\mathfrak{W}_2$, then $\mathfrak{W}_1 J_1 = \mathfrak{W}_2 J_2 = \mathfrak{B}'_1 \cap \mathfrak{B}'_2$ and $J_1 J_2^{-1} = J$.*

($J_1 J_2^{-1}$ means first J_1 and then J_2^{-1}. The term "contraction" is used in the usual sense; hence, if $W \in \mathfrak{W}_1$, then $WJ_1 = WI_1$ etc.) Observe that 8.3 says

that $W_1J_1 = W_2J_2$, where $W_1 \in \mathfrak{W}_1$ and $W_2 \in \mathfrak{W}_2$, if and only if $W_1J = W_2$, i.e., if and only if W_1 and W_2 correspond under J. Hence 8.3 states indeed that the interlacing of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ has been carried out by identifying the elements of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ which correspond under J.

DEFINITION 8.1. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules and let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Then, a module $\mathfrak{B}$ which satisfies conditions* 8.1, 8.2 *and* 8.3 *is said to have been obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$. We call $\mathfrak{W}_1$ and $\mathfrak{W}_2$ the "laces" and J the "lacing isomorphism" and say that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ were interlaced by lacing the laces $\mathfrak{W}_1$ and $\mathfrak{W}_2$ together according to the lacing isomorphism J.*

We now prove the following theorem.

THEOREM 8.1. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules and let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Then there always exists a module $\mathfrak{B}$ which is obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. This $\mathfrak{B}$ is unique except for isomorphisms. Conversely, if $\mathfrak{B}$ is a module and $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$ are two submodules of $\mathfrak{B}$ where $(\mathfrak{B}'_1, \mathfrak{B}'_2) = \mathfrak{B}$, then $\mathfrak{B}$ can always be obtained by interlacing any two modules $\mathfrak{B}_1$ and $\mathfrak{B}_2$ which are isomorphic with respectively $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$.*

Proof. We first prove the uniqueness of $\mathfrak{B}$. Hereto, let $\mathfrak{B}_1$, $\mathfrak{W}_1$, $\mathfrak{B}_2$, $\mathfrak{W}_2$, J and $\mathfrak{B}$ have the same meaning as in Theorem 8.1 and let I_1, $\mathfrak{B}'_1$, I_2, $\mathfrak{B}'_2$, J_1 and J_2 have the same meaning as in conditions 8.1, 8.2 and 8.3. Then, since $\mathfrak{B} = (\mathfrak{B}'_1, \mathfrak{B}'_2)$, every element V of $\mathfrak{B}$ can be written as $V = V_1I_1 + V_2I_2$, where $V_1 \in \mathfrak{B}_1$ and $V_2 \in \mathfrak{B}_2$. This representation of V is not unique, as the following statement indicates.

STATEMENT 8.1. *The equality $V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$, where $V_1, V'_1 \in \mathfrak{B}_1$ and $V_2, V'_2 \in \mathfrak{B}_2$, is equivalent to the following two conditions:*

8.4 $$V_1 - V'_1 \in \mathfrak{W}_1 \text{ and } V'_2 - V_2 \in \mathfrak{W}_2;$$

8.5 $$(V_1 - V'_1)J = V'_2 - V_2.$$

We prove statement 8.1 by observing that $V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$ is the same as $(V_1 - V'_1)I_1 = (V'_2 - V_2)I_2$. Since I_1 maps $\mathfrak{B}_1$ onto $\mathfrak{B}'_1$ and I_2 maps $\mathfrak{B}_2$ onto $\mathfrak{B}'_2$, we conclude that $(V_1 - V'_1)I_1 \in \mathfrak{B}'_1 \cap \mathfrak{B}'_2$ and $(V'_2 - V_2)I_2 \in \mathfrak{B}'_1 \cap \mathfrak{B}'_2$. Consequently, it follows from 8.3, that $V_1 - V'_1 \in \mathfrak{W}_1$ and $V'_2 - V_2 \in \mathfrak{W}_2$ which proves 8.4. We can then conclude that $(V_1 - V'_1)I_1 = (V_1 - V'_1)J_1$ and that $(V'_2 - V_2)I_2 = (V'_2 - V_2)J_2$, which implies that $(V_1 - V'_1)J_1 = (V'_2 - V_2)J_2$ and hence that $(V_1 - V'_1)J_1J_2^{-1} = V'_2 - V_2$. It then follows from condition 8.3 that $(V_1 - V'_1)J = V'_2 - V_2$ which proves 8.5. Conversely, suppose that 8.4 and 8.5 are satisfied. We can then write for 8.5, $(V_1 - V'_1) \cdot J_1J_2^{-1} = V'_2 - V_2$, hence $(V_1 - V'_1)J_1 = (V'_2 - V_2)J_2$, hence $(V_1 - V'_1)I_1 = (V'_2 - V_2)I_2$, which shows that $V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$; consequently, statement 8.1 is proved.

The uniqueness of $\mathfrak{B}$ now follows easily. Namely, suppose that $\mathfrak{B}^*$ is also obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing

isomorphism. Let I^*_1 and I^*_2 have the same meaning for $\mathfrak{B}^*$ as I_1 and I_2 have for $\mathfrak{B}$. We associate to the element $V_1I_1+ V_2I_2$ of $\mathfrak{B}$, where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, the element $V_1I^*_1+ V_2I^*_2$ of $\mathfrak{B}^*$ and we claim that this correspondence is an isomorphism from $\mathfrak{B}$ onto $\mathfrak{B}^*$. In the first place, it is an immediate consequence of statement 8.1 that $V_1I_1+ V_2I_2= V'_1I_1+ V'_2I_2$, where $V_1, V'_1 \in \mathfrak{V}_1$ and $V_2, V'_2 \in \mathfrak{V}_2$, if and only if $V_1I^*_1+ V_2I^*_2= V'_1I^*_1+ V'_2I^*_2$; consequently, the correspondence is well defined and one-one. The fact that this correspondence is an isomorphism (A-isomorphism (!)) then follows trivially and hence the uniqueness of $\mathfrak{B}$ is proved.

We now prove the existence of $\mathfrak{B}$. Hereto, let $\mathfrak{V}_1$, $\mathfrak{W}_1$, $\mathfrak{V}_2$, $\mathfrak{W}_2$ and J have the same meaning as in Theorem 8.1. We consider the pairs (V_1, V_2), where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, and define the following equivalence relation for them.

DEFINITION 8.2. *Two pairs* (V_1, V_2) *and* (V'_1, V'_2), *where* $V_1, V'_1 \in \mathfrak{V}_1$ *and* $V_2, V'_2 \in \mathfrak{V}_2$, *are equivalent if conditions* 8.4 *and* 8.5 *are satisfied:*

8.4 $$V_1- V'_1 \in \mathfrak{W}_1 \text{ and } V'_2- V_2 \in \mathfrak{W}_2.$$

8.5 $$(V_1- V'_1)J = V'_2- V_2.$$

(This definition of equivalence is indeed the natural one, as follows from statement 8.1 and the fact that the pair (V_1, V_2) corresponds to the element $V_1I_1+ V_2I_2$ of the module $\mathfrak{B}$ which we are trying to construct.) It can be proved without the slightest difficulty that the equivalence relation of definition 8.2 satisfies the conditions of reflexivity, symmetry and transitivity. The class of pairs which is equivalent to (V_1, V_2), where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, is denoted by $[V_1, V_2]$; hence $[V_1, V_2] = [V'_1, V'_2]$ if and only if conditions 8.4 and 8.5 are satisfied. *The elements of* $\mathfrak{B}$ *are defined as these classes* $[V_1, V_2]$ *of equivalent pairs. Addition of the elements* $[V_1, V_2]$ *and* $[V'_1, V'_2]$ *of* $\mathfrak{B}$ *and multiplication by elements* $\mathfrak{a} \in A$ *are defined by:*

8.6 $$[V_1, V_2] + [V'_1, V'_2] = [V_1+ V'_1, V_2+ V'_2].$$

8.7 $$[V_1, V_2]\,\mathfrak{a} = [V_1\mathfrak{a}, V_2\mathfrak{a}].$$

The proof that definitions 8.6 and 8.7 do not depend on the representatives which are used for the classes $[V_1, V_2]$ and $[V'_1, V'_2]$ and that $\mathfrak{B}$ is thus made into an A-module gives no difficulty whatsoever. We proceed to show that $\mathfrak{B}$ satisfies conditions 8.1, 8.2 and 8.3. Condition 8.1 is proved by observing that the mapping $V_1I_1= [V_1, 0_2]$, where $V_1 \in \mathfrak{V}_1$ and 0_2 is the zero element of $\mathfrak{V}_2$, is clearly an isomorphism from $\mathfrak{V}_1$ onto a submodule $\mathfrak{V}'_1$ of $\mathfrak{B}$; in the same way $V_2I_2= [0_1, V_2]$, where $V_2 \in \mathfrak{V}_2$ and 0_1 is the zero element of $\mathfrak{V}_1$, is an isomorphism from $\mathfrak{V}_2$ onto a submodule $\mathfrak{V}'_2$ of $\mathfrak{B}$. Condition 8.2 then follows from the fact that the arbitrary element $[V_1, V_2]$ of $\mathfrak{B}$ is the sum of V_1I_1 and V_2I_2:

$$[V_1, V_2] = [V_1, 0_2] + [0_1, V_2] = V_1I_1+ V_2I_2.$$

In order to prove 8.3, we observe that $\mathfrak{W}_1J_1$ consists of the elements $[W_1, 0_2]$, where $W_1 \in \mathfrak{W}_1$, and $\mathfrak{W}_2J_2$ of the elements $[0_1, W_2]$ where $W_2 \in \mathfrak{W}_2$. Since $[W_1, 0_2] = [0_1, W_1J]$ we conclude that $\mathfrak{W}_1J_1 \subseteq \mathfrak{W}_2J_2$, while $[0_1, W_2] = [W_2J^{-1}, 0_2]$ shows that $\mathfrak{W}_2J_2 \subseteq \mathfrak{W}_1J_1$; hence $\mathfrak{W}_1J_1= \mathfrak{W}_2J_2$. An element V of $\mathfrak{B}$ lies in

$\mathfrak{B}'_1 \cap \mathfrak{B}'_2$ if and only if it can be written as $V = [V_1, 0_2] = [0_1, V_2]$, where $V_1 \in \mathfrak{B}_1$ and $V_2 \in \mathfrak{B}_2$. This implies that $V_1 \in \mathfrak{W}_1$ and hence that $V \in \mathfrak{W}_1 J_1$; consequently, $\mathfrak{B}'_1 \cap \mathfrak{B}'_2 \subseteq \mathfrak{W}_1 J_1$. Conversely, if $V \in \mathfrak{W}_1 J_1$, then $V = [W_1, 0_2]$ where $W_1 \in \mathfrak{W}_1$. This already proves that $V \in \mathfrak{B}'_1$ and hence that $\mathfrak{W}_1 J_1 \subseteq \mathfrak{B}'_1$. Since, furthermore, $[W_1, 0_2] = [0_1, W_1 J] \in \mathfrak{B}'_2$, we conclude that $\mathfrak{W}_1 J_1 = \mathfrak{B}'_1 \cap \mathfrak{B}'_2$ and hence that $\mathfrak{W}_1 J_1 = \mathfrak{W}_2 J_2 = \mathfrak{B}'_1 \cap \mathfrak{B}'_2$. Finally, if $W_1 \in \mathfrak{W}_1$, then $W_1 J_1 J_2^{-1} = [W_1, 0_2] J_2^{-1} = [0_1, W_1 J] J_2^{-1} = W_1 J$ which proves that $J_1 J_2^{-1} = J$; hence 8.3 is fully proved. In order to prove the last part of Theorem 8.1, let $\mathfrak{B}$ be a module which has two submodules $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$ such that $\mathfrak{B} = (\mathfrak{B}'_1, \mathfrak{B}'_2)$. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two modules which are isomorphic with respectively $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$. Let I_1 be an isomorphism from $\mathfrak{B}_1$ onto $\mathfrak{B}'_1$ and I_2 an isomorphism from $\mathfrak{B}_2$ onto $\mathfrak{B}'_2$. Let $\mathfrak{W}_1$ be the submodule $(\mathfrak{B}'_1 \cap \mathfrak{B}'_2) I_1^{-1}$ of $\mathfrak{B}_1$ and let $\mathfrak{W}_2$ be the submodule $(\mathfrak{B}'_1 \cap \mathfrak{B}'_2) I_2^{-1}$ of $\mathfrak{B}_2$. Let J_1 be the contraction of I_1 on $\mathfrak{W}_1$ and J_2 the contraction of I_2 on $\mathfrak{W}_2$. Since J_1 and J_2 map respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ onto $\mathfrak{B}'_1 \cap \mathfrak{B}'_2$, the isomorphism $J = J_1 J_2^{-1}$ is a well-defined isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. We claim that if we interlace $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism, we obtain a module $\mathfrak{B}^*$ which is isomorphic with $\mathfrak{B}$. We can consider the elements of $\mathfrak{B}^*$, as above, as the classes $[V_1, V_2]$ of equivalent pairs, where $V_1 \in \mathfrak{B}_1$ and $V_2 \in \mathfrak{B}_2$. If then we associate to the element $[V_1, V_2]$ of $\mathfrak{B}^*$ the element $V_1 I_1 + V_2 I_2$ of $\mathfrak{B}$, we clearly obtain an isomorphism (A-isomorphism!) from $\mathfrak{B}^*$ onto $\mathfrak{B}$. This completes the proof of Theorem 8.1.

REMARK 8.1. If we interlace two modules $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using the zero modules of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ as laces, we obtain the usual direct sum of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Hence the interlacing of modules can be considered as a generalization of the direct sum of modules.

The following statement and lemma will be used in the next section.

STATEMENT 8.2. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two A-modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules. Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be A-isomorphic and let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Let $\mathfrak{B}$ be obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. Then, if I_1, I_2, $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$ have the same meaning as in conditions* 8.1, 8.2 *and* 8.3 *and if V_1 and V_2 denote elements of respectively $\mathfrak{B}_1$ and $\mathfrak{B}_2$:*

8.8 $V_1 I_1 + V_2 I_2 \in \mathfrak{B}'_1$ *if and only if* $V_2 \in \mathfrak{W}_2$;
$V_1 I_1 + V_2 I_2 \in \mathfrak{B}'_2$ *if and only if* $V_1 \in \mathfrak{W}_1$.

8.9 $V_1 I_1 + V_2 I_2 \in \mathfrak{B}'_1 \cap \mathfrak{B}'_2$ *if and only if*
$V_1 \in \mathfrak{W}_1$ *and* $V_2 \in \mathfrak{W}_2$.

Proof. If $V_1 I_1 + V_2 I_2 \in \mathfrak{B}'_1$, then $V_1 I_1 + V_2 I_2 = V'_1 I_1 + 0_2 I_2$, where $V'_1 \in \mathfrak{B}_1$ and 0_2 is the zero element of $\mathfrak{B}_2$. Statement 8.1 then implies that $V_2 \in \mathfrak{W}_2$. Conversely, if $V_2 \in \mathfrak{W}_2$, that same statement implies that $V_1 I_1 + V_2 I_2 = (V_1 + V_2 J^{-1}) I_1 + 0_2 I_2 \in \mathfrak{B}'_1$ and hence the first part of 8.8 is proved. The second part of 8.8 is proved in the same way while 8.9 is an immediate consequence of 8.8.

LEMMA 8.1. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two A-modules with respectively $\mathfrak{a}_1$ and $\mathfrak{a}_2$ as annihilators. Let $\mathfrak{W}_1$ be a submodule of $\mathfrak{B}_1$ and $\mathfrak{W}_2$ a submodule of $\mathfrak{B}_2$, where $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are A-isomorphic. Let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. The module $\mathfrak{B}$, which is obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$ using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism, has $\mathfrak{a}_1 \cap \mathfrak{a}_2$ as annihilating ideal. If furthermore $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have respective finite lengths l_1 and l_2 and hence $\mathfrak{W}_1$ and $\mathfrak{W}_2$ have the common finite length, say λ, then $\mathfrak{B}$ has finite length $l = l_1 + l_2 - \lambda$.*

Proof. (A module is said to have finite length if it has a composition series of finite length.) We know that $\mathfrak{B} = (\mathfrak{B}'_1, \mathfrak{B}'_2)$ and hence that the annihilating ideal $\mathfrak{a}$ of $\mathfrak{B}$ is the intersection of the annihilating ideal $\mathfrak{a}'_1$ of $\mathfrak{B}'_1$ and $\mathfrak{a}'_2$ of $\mathfrak{B}'_2$. The fact that $\mathfrak{B}'_1$ is isomorphic with $\mathfrak{B}_1$ and $\mathfrak{B}'_2$ with $\mathfrak{B}_2$ implies that $\mathfrak{a}'_1 = \mathfrak{a}_1$ and $\mathfrak{a}'_2 = \mathfrak{a}_2$ and hence that $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$. Now suppose that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have respective finite lengths l_1 and l_2 and that the common length of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ is λ. Then $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$ have respective finite lengths l_1 and l_2 and, since property 8.3 implies that $\mathfrak{B}'_1 \cap \mathfrak{B}'_2$ is isomorphic with $\mathfrak{W}_1$ and $\mathfrak{W}_2$, the module $\mathfrak{B}'_1 \cap \mathfrak{B}'_2$ has finite length λ. The difference module $\mathfrak{B} - \mathfrak{B}'_1$ satisfies the relation $\mathfrak{B} - \mathfrak{B}'_1 = (\mathfrak{B}'_1, \mathfrak{B}'_2) - \mathfrak{B}'_1 = \mathfrak{B}'_2 - \mathfrak{B}'_1 \cap \mathfrak{B}'_2$, which implies that $\mathfrak{B}$ has finite length l, where $l - l_1 = l_2 - \lambda$ and hence $l = l_1 + l_2 - \lambda$. This completes the proof of lemma 8.1.

REMARK 8.2. It is interesting to note that lemma 8.1 implies that the annihilating ideal of the module $\mathfrak{B}$, obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, does not depend on the laces and lacing isomorphism used; it depends only on the annihilating ideals of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. In particular, lacing always produces a faithful representation from two faithful representations. The length of $\mathfrak{B}$ depends on the length of the laces, but not on the lacing isomorphism.

In the next section we investigate only those properties of interlacing which are needed for the proof of Theorem 9.1. The author believes, however, that a systematic investigation of interlacing would be worth while.

9. The faithful, completely indecomposable representation space of a completely primary ring. We now return to the study of modules with commutative operator domains. All notations and conventions will be the same as in the first seven sections of this paper. Hence the term module will be used only for A-module where A is always a commutative ring with unit element and where condition I-1 of the introduction is assumed to hold. Isomorphism always means A-isomorphism and, as before, the radical of the annihilating ideal of a module is called the radical of the module. The following corollary is an immediate consequence of lemma 8.1 and the fact that the radical of the intersection of two ideals is the intersection of the radicals of these ideals.

COROLLARY 9.1. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two A-modules with respectively $\mathfrak{r}_1$ and $\mathfrak{r}_2$ as radicals. Then, if $\mathfrak{B}$ is obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, no matter what laces and lacing isomorphism are used, the radical of $\mathfrak{B}$ is $\mathfrak{r}_1 \cap \mathfrak{r}_2$.*

Since the proof of Theorem 9.1 depends on interlacing by means of lacing isomorphisms which are not "extendable," we discuss the notion of extendable isomorphisms in the following definition and lemma. We remind the reader of the convention that the symbol $\subset$ is used exclusively for *proper* inclusion.

DEFINITION 9.1. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two A-modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules. Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be A-isomorphic and J an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Then J is called extendable if the following two conditions are satisfied.*

9.1 *There exist submodules $\mathfrak{W}'_1$ and $\mathfrak{W}'_2$ of respectively $\mathfrak{B}_1$ and $\mathfrak{B}_2$, where $\mathfrak{W}_1 \subset \mathfrak{W}'_1$ and $\mathfrak{W}_2 \subset \mathfrak{W}'_2$,*

9.2 *J can be extended to an isomorphism J' from $\mathfrak{W}'_1$ onto $\mathfrak{W}'_2$.*

It is clear that, if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are not-isomorphic A-modules with $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as respective *maximal* submodules, then any isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$ is not extendable.

LEMMA 9.1. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two A-modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules. Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be A-isomorphic and J an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Then J is extendable if and only if there exist elements $V_1 \in \mathfrak{B}_1$ and $V_2 \in \mathfrak{B}_2$, where:*

9.3 V_1 not $\in \mathfrak{W}_1$ *and* V_2 not $\in \mathfrak{W}_2$.

9.4 $\mathfrak{W}_1 : V_1 = \mathfrak{W}_2 : V_2$; *we denote this ideal by* $\mathfrak{c}$.

9.5 $(V_1\gamma)J = V_2\gamma$ *for all* $\gamma \in \mathfrak{c}$.

Proof. (The quotient of modules is defined as before. The quotient $\mathfrak{W}_1 : V_1$ of a module $\mathfrak{W}_1$ by an element V_1 is defined as the quotient of $\mathfrak{W}_1$ by the cyclic module generated by V_1. Consequently, $V_1\gamma \in \mathfrak{W}_1$ and $V_2\gamma \in \mathfrak{W}_2$ for all $\gamma \in \mathfrak{c}$, and hence condition 9.5 makes sense.) Let $\mathfrak{B}_1$, $\mathfrak{W}_1$, $\mathfrak{B}_2$, $\mathfrak{W}_2$ and J have the same meaning as in lemma 9.1 where J is extendable. Let $\mathfrak{W}'_1$, $\mathfrak{W}'_2$ and J' have the same meaning as in definition 9.1. Then, because of condition 9.1, we can choose an element $V_1 \in \mathfrak{W}'_1$ where V_1 not $\in \mathfrak{W}_1$. We denote V_1J' by V_2 and claim that V_1 and V_2 satisfy conditions 9.3, 9.4 and 9.5. In the first place J' is an extension of J and $\mathfrak{W}_1J = \mathfrak{W}_2$; hence V_1 not $\in \mathfrak{W}_1$ implies V_2 not $\in \mathfrak{W}_2$ and 9.3 is satisfied. Secondly the module $(\mathfrak{W}_1, V_1)$, generated by $\mathfrak{W}_1$ and V_1, is mapped isomorphically onto $(\mathfrak{W}_2, V_2)$ by J' while $\mathfrak{W}_1J' = \mathfrak{W}_1J = \mathfrak{W}_2$. This implies, since we are dealing with operator isomorphisms, that $\mathfrak{W}_1 : (\mathfrak{W}_1, V_1) = \mathfrak{W}_2 : (\mathfrak{W}_2, V_2)$. This, however, is exactly condition 9.4 since $\mathfrak{W}_1 : (\mathfrak{W}_1, V_1) = \mathfrak{W}_1 : \mathfrak{W}_1 \cap \mathfrak{W}_1 : V_1 = \mathfrak{B}_1 \cap \mathfrak{W}_1 : V_1 = \mathfrak{W}_1 : V_1$ and in the same way $\mathfrak{W}_2 : (\mathfrak{W}_2, V_2) = \mathfrak{W}_2 : V_2$. Finally, since J' is an operator isomorphism, $(V_1\gamma)J' = (V_1J')\gamma = V_2\gamma$. If furthermore $\gamma \in \mathfrak{c} = \mathfrak{W}_1 : V_1$, then $V_1\gamma \in \mathfrak{W}_1$ and hence $(V_1\gamma)J' = (V_1\gamma)J$, from which we conclude that $(V_1\gamma)J = V_2\gamma$, which proves 9.5. Conversely, let $\mathfrak{B}_1$, $\mathfrak{W}_1$, $\mathfrak{B}_2$, $\mathfrak{W}_2$ again have the same meaning as above where now J is any A-isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. We assume that there exist elements $V_1 \in \mathfrak{B}_1$ and $V_2 \in \mathfrak{B}_2$ which satisfy conditions 9.3, 9.4 and 9.5 and prove that then J is extendable. We choose $\mathfrak{W}'_1 = (\mathfrak{W}_1, V_1)$ and $\mathfrak{W}'_2 = (\mathfrak{W}_2, V_2)$ and con-

sequently, as follows from 9.3, condition 9.1 is satisfied. Furthermore, J can be extended to an isomorphism J' from $\mathfrak{W}'_1$ onto $\mathfrak{W}'_2$ by defining:

$$(W_1 + V_1\mathfrak{a})J' = W_1J + V_2\mathfrak{a}, \text{ for } W_1 \in \mathfrak{W}_1 \text{ and } \mathfrak{a} \in A.$$

In order to show that this mapping J' is well defined, let $W_1 + V_1\mathfrak{a} = W'_1 + V_1\mathfrak{a}'$, where $W_1, W'_1 \in \mathfrak{W}_1$ and $\mathfrak{a}, \mathfrak{a}' \in A$. Then $W_1 - W'_1 = V_1(\mathfrak{a}' - \mathfrak{a})$ and consequently $\mathfrak{a}' - \mathfrak{a} \in \mathfrak{W}_1 : V_1 = \mathfrak{c}$. Condition 9.4 then gives that $V_2(\mathfrak{a}' - \mathfrak{a}) \in \mathfrak{W}_2$ and 9.5 that $(V_1(\mathfrak{a}' - \mathfrak{a}))J = V_2(\mathfrak{a}' - \mathfrak{a})$. This proves that $W_1J - W'_1J = (W_1 - W'_1)J = V_2\mathfrak{a}' - V_2\mathfrak{a}$ and hence that $W_1J + V_2\mathfrak{a} = W'_1J + V_2\mathfrak{a}'$; consequently, J' is well defined. It is trivial to show that J' is a homomorphism from $\mathfrak{W}'_1$ onto $\mathfrak{W}'_2$. We now prove that the zero element 0_1 of $\mathfrak{W}'_1$ is the only element of $\mathfrak{W}'_1$ which is mapped by J' on the zero element 0_2 of $\mathfrak{W}'_2$. Hereto, let $(W_1 + V_1\mathfrak{a})J' = 0_2$, where $W_1 \in \mathfrak{W}_1$ and $\mathfrak{a} \in A$. Then $W_1J + V_2\mathfrak{a} = 0_2$ and hence $W_1J = -V_2\mathfrak{a}$ which shows that $\mathfrak{a} \in \mathfrak{W}_2 : V_2 = \mathfrak{c}$. It then follows from conditions 9.4 and 9.5 that $V_1\mathfrak{a} \in \mathfrak{W}_1$ and $(V_1\mathfrak{a})J = V_2\mathfrak{a}$. We conclude that $W_1J + (V_1\mathfrak{a})J = (W_1 + V_1\mathfrak{a})J = 0_2$ and hence, since J is an isomorphism, that $W_1 + V_1\mathfrak{a} = 0_1$. This completes the proof of lemma 9.1.

The usefulness of the notion of extendable isomorphism for interlacing is apparent from the following lemma and its corollaries.

LEMMA 9.2. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two A-modules with the same radical $\mathfrak{p}$, where $\mathfrak{p}$ is a maximal prime ideal of A. Let 0_1 and 0_2 be the zero elements of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ and let $\mathfrak{W}_1$ be a submodule of $\mathfrak{B}_1$ and $\mathfrak{W}_2$ a submodule of $\mathfrak{B}_2$, where $0_1 : \mathfrak{p} \subseteq \mathfrak{W}_1$ and $0_2 : \mathfrak{p} \subseteq \mathfrak{W}_2$. Let J be a not-extendable isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$ and let $\mathfrak{B}$ be obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. Then, if 0 is the zero element of $\mathfrak{B}$ and if $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$ have the customary meaning of definitions* 8.1, 8.2 *and* 8.3:

9.6 *The radical of $\mathfrak{B}$ is $\mathfrak{p}$.*

9.7 $0 : \mathfrak{p} \subseteq \mathfrak{B}'_1 \cap \mathfrak{B}'_2$.

9.8 *The three modules* $0 : \mathfrak{p}$, $0_1 : \mathfrak{p}$ *and* $0_2 : \mathfrak{p}$ *are isomorphic.*

Proof. It is *always* true, according to corollary 9.1, that, if two modules with the same radical $\mathfrak{p}$ are interlaced, the resulting module has as radical $\mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}$; hence 9.6 is proved. Furthermore, since J is an operator isomorphism and since $0_1 : \mathfrak{p} \subseteq \mathfrak{W}_1$ and $0_2 : \mathfrak{p} \subseteq \mathfrak{W}_2$, the module $0_1 : \mathfrak{p}$ is mapped by J onto $0_2 : \mathfrak{p}$ and hence $0_1 : \mathfrak{p}$ is isomorphic with $0_2 : \mathfrak{p}$. In order to prove 9.7 and to prove that $0_1 : \mathfrak{p}$ is isomorphic with $0 : \mathfrak{p}$, let I_1, I_2, J_1 and J_2 have the meaning of definitions 8.1, 8.2 and 8.3. Hence $\mathfrak{B}_1I_1 = \mathfrak{B}'_1$ and $\mathfrak{B}_2I_2 = \mathfrak{B}'_2$ and the elements of $\mathfrak{B}$ can be written as $V_1I_1 + V_2I_2$, where $V_1 \in \mathfrak{B}_1$ and $V_2 \in \mathfrak{B}_2$. We now prove the following statement.

STATEMENT 9.1. *The element $V_1I_1 + V_2I_2$ of $\mathfrak{B}$, where $V_1 \in \mathfrak{B}_1$ and $V_2 \in \mathfrak{B}_2$, belongs to $0 : \mathfrak{p}$ if and only if the following two conditions are satisfied:*

9.9 $V_1 \in \mathfrak{W}_1$ *and* $V_2 \in \mathfrak{W}_2$.

9.10 $V_1J + V_2 \in 0_2 : \mathfrak{p}$.

Suppose that conditions 9.9 and 9.10 are satisfied. Then, for any $\pi \in \mathfrak{p}$, cer-

tainly $V_1\pi \in \mathfrak{B}_1$ and $-V_2\pi \in \mathfrak{B}_2$ and $(V_1\pi)J = -V_2\pi$. This is, according to statement 8.1, equivalent with $(V_1I_1 + V_2I_2)\pi = 0_1I_1 + 0_2I_2 = 0$ and hence $V_1I_1 + V_2I_2 \in 0 : \mathfrak{p}$. Conversely, let $V_1I_1 + V_2I_2 \in 0 : \mathfrak{p}$; i.e., for any $\pi \in \mathfrak{p}$, $(V_1\pi)I_1 + (V_2\pi)I_2 = 0_1I_1 + 0_2I_2$. Again, this is equivalent with $V_1\pi \in \mathfrak{B}_1$ and $-V_2\pi \in \mathfrak{B}_2$ and $(V_1\pi)J = -V_2\pi$ for all $\pi \in \mathfrak{p}$. Hence we have the following statement.

STATEMENT 9.2. *If* $V_1I_1 + V_2I_2 \in 0 : \mathfrak{p}$, *then*

9.11 $\qquad V_1 \in \mathfrak{B}_1 : \mathfrak{p}$ *and* $V_2 \in \mathfrak{B}_2 : \mathfrak{p}$;

9.12 $\qquad (V_1\pi)J = -V_2\pi$ *for all* $\pi \in \mathfrak{p}$.

We first conclude that either condition 9.9 is satisfied or *simultaneously* V_1 not $\in \mathfrak{B}_1$ and V_2 not $\in \mathfrak{B}_2$. Hereto, let $V_1 \in \mathfrak{B}_1$ which implies that V_1J is defined and hence that $(V_1\pi)J = (V_1J)\pi$. Condition 9.12 then states that $(V_1J + V_2)\pi = 0_2$ for all $\pi \in \mathfrak{p}$, i.e., that $V_1J + V_2 \in 0_2 : \mathfrak{p}$. Consequently, since $0_2 : \mathfrak{p} \subseteq \mathfrak{B}_2$ and $V_1J \in \mathfrak{B}_2$, we see that $V_2 \in \mathfrak{B}_2$. We show, in the same way, that if $V_2 \in \mathfrak{B}_2$ then $V_1 \in \mathfrak{B}_1$ and hence the conclusion is proved. Secondly, we show that V_1 not $\in \mathfrak{B}_1$ and V_2 not $\in \mathfrak{B}_2$ contradicts the hypothesis that J is not extendable. If V_1 not $\in \mathfrak{B}_1$, then $\mathfrak{B}_1 : V_1 \neq A$, while 9.11 implies that $\mathfrak{p} \subseteq \mathfrak{B}_1 : V_1$; hence, since $\mathfrak{p}$ is maximal, $\mathfrak{p} = \mathfrak{B}_1 : V_1$. In the same way, V_2 not $\in \mathfrak{B}_2$ implies that $\mathfrak{p} = \mathfrak{B}_2 : V_2$. Consequently, if V_1 not $\in \mathfrak{B}_1$ and V_2 not $\in \mathfrak{B}_2$, conditions 9.3, 9.4 and 9.5 are satisfied for $\mathfrak{B}_1$, $\mathfrak{B}_2$, V_1, $-V_2$ and J; and J is extendable. This proves that 9.9 is satisfied while, as was pointed out above, 9.12 becomes 9.10 when 9.9 is satisfied. Hence statement 9.1 is completely proved. It then immediately follows from conditions 9.9 and 8.9 that $0 : \mathfrak{p} \subseteq \mathfrak{B}'_1 \cap \mathfrak{B}'_2$. Hence 9.7 is proved and all there remains to be shown is that $0_1 : \mathfrak{p}$ is isomorphic with $0 : \mathfrak{p}$. We prove this fact by showing that I_1 maps $0_1 : \mathfrak{p}$ onto $0 : \mathfrak{p}$; i.e., that $(0_1 : \mathfrak{p})I_1 = 0 : \mathfrak{p}$. If $V \in 0 : \mathfrak{p}$, then $V = V_1I_1 + V_2I_2$ where 9.9 and 9.10 are satisfied. Then V_2J^{-1} is defined and hence $V_1I_1 + V_2I_2 = (V_1 + V_2J^{-1})I_1 + 0_2I_2$. Condition 9.10 states that $V_1 + V_2J^{-1} \in 0_1 : \mathfrak{p}$ and hence $V \in (0_1 : \mathfrak{p})I_1$ which proves that $0 : \mathfrak{p} \subseteq (0_1 : \mathfrak{p})I_1$. Conversely, if $V = V_1I_1 + V_2I_2 \in (0_1 : \mathfrak{p})I_1$, where $V_1 \in \mathfrak{B}_1$ and $V_2 \in \mathfrak{B}_2$, then $V_1I_1 + V_2I_2 = V'_1I_1 + 0_2I_2$, where $V'_1 \in 0_1 : \mathfrak{p}$. This implies that $V_1 - V'_1 \in \mathfrak{B}_1$ and $-V_2 \in \mathfrak{B}_2$ and $(V_1 - V'_1)J = -V_2$. Since $V'_1 \in 0_1 : \mathfrak{p} \subseteq \mathfrak{B}_1$, we see that $V_1 \in \mathfrak{B}_1$ and 9.9 is proved. Furthermore, $(V_1 - V'_1)J = -V_2$ implies that $V_1J + V_2 = V'_1J$, while $V'_1 \in 0_1 : \mathfrak{p}$ implies that $V'_1J \in 0_2 : \mathfrak{p}$. Hence 9.10 is satisfied, which shows that $(0_1 : \mathfrak{p})I_1 \subseteq 0 : \mathfrak{p}$ and hence that $(0_1 : \mathfrak{p})I_1 = 0 : \mathfrak{p}$. This completes the proof of lemma 9.2.

If $\mathfrak{B}$ is an A-module of finite length with radical $\mathfrak{p}$ and zero element 0, the length of $0 : \mathfrak{p}$ is usually referred to as the first Loewy invariant of $\mathfrak{B}$. The importance of lemma 9.2 lies in the fact that it states a condition for the invariance of this Loewy invariant under interlacing. The following immediate corollary of lemma 9.2 formulates this invariance explicitly.

COROLLARY 9.2. *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be two* A*-modules of finite length. Let* $\mathfrak{B}$ *be obtained by interlacing* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *where the radicals of* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$, *the laces,*

and the lacing isomorphism satisfy the conditions of lemma 9.2. *Then,* $\mathfrak{B}_1$, $\mathfrak{B}_2$ *and* $\mathfrak{B}$ *have the same first Loewy invariant and the same radical.*

We know that a module $\mathfrak{B}$ of finite length is completely indecomposable if and only if its first Loewy invariant is equal to one. Furthermore, the radical $\mathfrak{p}$ of a completely indecomposable module $\mathfrak{B}$ is always a maximal ideal and every non-zero submodule of $\mathfrak{B}$ always contains $0 : \mathfrak{p}$. This proves the following simple and important corollary on interlacing of completely indecomposable modules.

COROLLARY 9.3. *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be two completely indecomposable* A*-modules with the same radical* $\mathfrak{p}$. *Let* $\mathfrak{B}$ *be obtained by interlacing* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$, *using non-zero laces and a not-extendable lacing isomorphism. Then* $\mathfrak{B}$ *is completely indecomposable and has radical* $\mathfrak{p}$.

REMARK 9.1. We can see from the following that we can not omit the condition that the lacing isomorphism is not extendable from corollary 9.3 (and hence not from corollary 9.2 and lemma 9.2). Choose for the $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of corollary 9.3 one and the same completely indecomposable A-module $\mathfrak{B}_1$ of length $l_1 > 1$ and with $\mathfrak{q}_1$ as annihilating ideal; hence $\mathfrak{B}_1 = \mathfrak{B}_2$ and, according to lemma 2.1, the length of $\mathfrak{q}_1$ is also l_1. Let $\mathfrak{W}_1$ be a maximal submodule of $\mathfrak{B}_1$ and let J be an A-automorphism from $\mathfrak{W}_1$ onto itself; then $\mathfrak{W}_1$ has length $l_1 - 1 > 0$ and hence is not the zero module of $\mathfrak{B}_1$ and, according to corollary 3.1, J is extendable to an automorphism from $\mathfrak{B}_1$ onto $\mathfrak{B}_1$. Let $\mathfrak{B}$ be obtained by interlacing $\mathfrak{B}_1$ with itself, using $\mathfrak{W}_1$ as both laces and J as lacing isomorphism. Then, all the conditions of corollary 9.3 are satisfied except that J is extendable, while we can see as follows that $\mathfrak{B}$ is not completely indecomposable. According to lemma 8.1, the length of $\mathfrak{B}$ is $l_1 + l_1 - (l_1 - 1) = l_1 + 1$ and the annihilating ideal of $\mathfrak{B}$ is $\mathfrak{q}_1 \cap \mathfrak{q}_1 = \mathfrak{q}_1$. Consequently, since $\mathfrak{q}_1$ has length l_1, it follows from lemma 2.1 that $\mathfrak{B}$ is not completely indecomposable.

We have now developed the theory of interlacing far enough to prove Theorem 9.1. Before doing this, however, we give one example of interlacing in order to demonstrate the power of this composition for the construction of counterexamples.

EXAMPLE 9.1. We know that the length of any completely indecomposable module and of any cyclic module is always equal to the length of its annihilator. We counterexample the converse of this statement by constructing an A-module $\mathfrak{B}$ which has the following properties.

9.13 $\mathfrak{B}$ satisfies the properties I-1, I-2 and I-3 of the introduction.

9.14 The zero element of $\mathfrak{B}$ is a primary module of $\mathfrak{B}$ and the radical of $\mathfrak{B}$ is a maximal ideal of A.

9.15 $\mathfrak{B}$ is indecomposable.

9.16 The length of $\mathfrak{B}$ is equal to the length of its annihilator.

9.17 $\mathfrak{B}$ is neither completely indecomposable nor cyclic.

We do this by choosing a Noetherian ring A with unit element (i.e., a commu-

tative ring with unit element whose ideals satisfy the ascending chain condition) and two ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ in A which have the following properties.

9.18 $\mathfrak{q}_1 \neq \mathfrak{q}_2$ and $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are primary ideals with the same maximal associated prime $\mathfrak{p}$.

9.19 The difference modules $\mathfrak{B}_1 = A - \mathfrak{q}_1$ and $\mathfrak{B}_2 = A - \mathfrak{q}_2$ both have finite length equal to 3.

9.20 $\mathfrak{q}_1 : \mathfrak{p} = \mathfrak{q}_2 : \mathfrak{p} = \mathfrak{p}$.

9.21 $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \mathfrak{p}^2$ and the length of the difference module $A - \mathfrak{p}^2$ is equal to 4.

We then have in $\mathfrak{B}_1$ and $\mathfrak{B}_2$ two A-modules of finite length 3 with the same maximal radical $\mathfrak{p}$, while $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the annihilating ideals of respectively $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Consequently, since $\mathfrak{q}_1 \neq \mathfrak{q}_2$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are not isomorphic. Furthermore, if 0_1 and 0_2 denote the zero elements of respectively $\mathfrak{B}_1$ and $\mathfrak{B}_2$, property 9.20 implies that both modules $\mathfrak{W}_1 = 0_1 : \mathfrak{p}$ and $\mathfrak{W}_2 = 0_2 : \mathfrak{p}$ have length 2; hence the first Loewy invariant of both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is 2 and $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are maximal submodules of respectively $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Furthermore, $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are isomorphic since each of these modules is isomorphic with the direct sum $A - \mathfrak{p} \dotplus A - \mathfrak{p}$. Let J be any fixed isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$; then, according to the sentence following definition 9.1, J is not extendable. We claim that the module $\mathfrak{B}$ which is obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism, has the required properties. In the first place, it follows from lemma 8.1, that $\mathfrak{B}$ has length $3 + 3 - 2 = 4$ and that the annihilating ideal of $\mathfrak{B}$ is $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \mathfrak{p}^2$ which, according to 9.21, also has length 4; hence 9.13 and 9.16 are satisfied. Furthermore, according to corollary 9.1, the radical of $\mathfrak{B}$ is equal to $\mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}$ which, since $\mathfrak{p}$ is maximal, implies that the zero module of $\mathfrak{B}$ is primary and hence 9.14 is proved. In the second place, since the conditions of corollary 9.2 are satisfied, the first Loewy invariant of $\mathfrak{B}$ is 2 and hence $\mathfrak{B}$ is not completely indecomposable. If $\mathfrak{B}$ were cyclic, $\mathfrak{B}$ would be A-isomorphic with the difference module $A - \mathfrak{p}^2$. This is not possible since $\mathfrak{B}$ contains two distinct maximal submodules $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$ which are isomorphic with respectively $A - \mathfrak{q}_1$ and $A - \mathfrak{q}_2$, while $A - \mathfrak{p}^2$ has the unique maximal submodule $\mathfrak{p} - \mathfrak{p}^2$; hence 9.17 is proved. In order to prove 9.15, suppose that $\mathfrak{B} = \mathfrak{A}_1 \dotplus \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are non-zero submodules of $\mathfrak{B}$. Then, either $\mathfrak{A}_1$ has length 1 and $\mathfrak{A}_2$ has length 3, or both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have length 2. We can easily see that in both cases, since $\mathfrak{A}_1 \cap \mathfrak{A}_2 = 0$, the modules $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have to be completely indecomposable. Hence, if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ denote the annihilating ideals of respectively $\mathfrak{A}_1$ and $\mathfrak{A}_2$, the lengths of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are equal to those of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. This, together with the facts that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are primary ideals with $\mathfrak{p}$ as associated prime and that $\mathfrak{p}^2 = \mathfrak{a}_1 \cap \mathfrak{a}_2$, leads easily to a contradiction; hence 9.15 is proved. We can choose for A the ring $A = K[x, y, z]$ which consists of the polynomials in three variables x, y and z with coefficients in the field K; and for $\mathfrak{q}_1$ and $\mathfrak{q}_2$ we can choose respectively the primary ideals $\mathfrak{q}_1 = (x, y^2, yz, z^2)$ and $\mathfrak{q}_2 = (x^2, y, xz, z^2)$ of A. Then,

x not $\in \mathfrak{q}_2$ and hence $\mathfrak{q}_1 \neq \mathfrak{q}_2$, while the common maximal associated prime of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ is $\mathfrak{p} = (x, y, z)$; hence 9.18 is satisfied. We easily prove that $\mathfrak{q}_1 \cap \mathfrak{q}_2 = (x^2, y^2, z^2, xy, xz, yz) = \mathfrak{p}^2$ and that the following two sequences are both composition series from $\mathfrak{p}^2$ to A:

$$\mathfrak{p}^2 \subset \mathfrak{q}_1 \subset (x, y, z^2) \subset \mathfrak{p} \subset A,$$
$$\mathfrak{p}^2 \subset \mathfrak{q}_2 \subset (x, y, z^2) \subset \mathfrak{p} \subset A.$$

This implies both 9.21 and 9.19. Finally, since $\mathfrak{p}^2 \subset \mathfrak{q}_1$, we have $\mathfrak{p} \subseteq \mathfrak{q}_1 : \mathfrak{p}$, while $\mathfrak{q}_1 : \mathfrak{p} \subseteq \mathfrak{p}$ follows from the fact that $\mathfrak{q}_1 : \mathfrak{p}$ is primary and has $\mathfrak{p}$ as associated prime. Hence $\mathfrak{p} = \mathfrak{q}_1 : \mathfrak{p}$ and, in the same way, $\mathfrak{p} = \mathfrak{q}_2 : \mathfrak{p}$, which proves 9.20 and completes example 9.1.

The proof of Theorem 9.1 uses the following simple fact of Noetherian rings A with unit element. (If $\mathfrak{q}$ is an ideal of A, a proper divisor $\mathfrak{q}_1$ of $\mathfrak{q}$ is of course called a minimal divisor of $\mathfrak{q}$, if there exists no ideal $\mathfrak{a}$ such that $\mathfrak{q} \subset \mathfrak{a} \subset \mathfrak{q}_1$.)

LEMMA 9.3. *Let A be a Noetherian ring with unit element and let $\mathfrak{q}$ be a primary ideal of A with maximal associated prime ideal $\mathfrak{p}$. Then, either $\mathfrak{q}$ is intersection-irreducible, or $\mathfrak{q}$ has two distinct minimal divisors $\mathfrak{q}_1$ and $\mathfrak{q}_2$. In the latter case, $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are always primary ideals with $\mathfrak{p}$ as associated prime, while $\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2$; furthermore, $(\mathfrak{q}_1, \mathfrak{q}_2)$ is the a common minimal divisor of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ and is also primary with $\mathfrak{p}$ as associated prime.*

Proof. We know that, since $\mathfrak{p}$ is maximal, every divisor of $\mathfrak{q}$, except A, is primary and has $\mathfrak{p}$ as associated prime and that $\mathfrak{B} = A - \mathfrak{q}$ has finite length. If $\mathfrak{q}$ is not intersection-irreducible, $\mathfrak{B}$ is not completely indecomposable and hence $\mathfrak{B}$ has then at least two distinct minimal submodules $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Consequently, $\mathfrak{B}_1 \cap \mathfrak{B}_2 = 0$ and $(\mathfrak{B}_1, \mathfrak{B}_2)$ is a common minimal divisor of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. The lemma then follows immediately from considering the ideals $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $(\mathfrak{q}_1, \mathfrak{q}_2)$ which are mapped onto respectively $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $(\mathfrak{B}_1, \mathfrak{B}_2)$ by the natural operator-homomorphism from A onto $A - \mathfrak{q}$.

THEOREM 9.1. *Let A be a Noetherian ring with unit element. Let $\mathfrak{q}$ be a primary ideal of A with maximal associated prime ideal $\mathfrak{p}$. Then there exists a completely indecomposable A-module $\mathfrak{B}$ with $\mathfrak{q}$ as annihilator.*

Proof. If the length of $\mathfrak{q}$ is 1, then $\mathfrak{q} = \mathfrak{p}$ and $A - \mathfrak{q}$ is a completely indecomposable A-module with $\mathfrak{q}$ as annihilator. Hence we can make the induction hypothesis that Theorem 9.1 has been proved when $\mathfrak{q}$ has length $1, 2, \ldots, l-1$. (The length of $\mathfrak{q}$ is the length of $A - \mathfrak{q}$.) Suppose that $\mathfrak{q}$ has length l. If $\mathfrak{q}$ is intersection-irreducible, $A - \mathfrak{q}$ is a completely indecomposable A-module (see example 7.2) with $\mathfrak{q}$ as annihilator. If $\mathfrak{q}$ is not intersection-irreducible then, according to lemma 9.3, $\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2$ where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are primary ideals with $\mathfrak{p}$ as associated prime and $l - 1$ as length. The induction hypothesis then guarantees the existence of two completely indecomposable A-modules $\mathfrak{B}_1$ and $\mathfrak{B}_2$ with $\mathfrak{q}_1$ and $\mathfrak{q}_2$ as respective annihilators. The modules $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have the same radical $\mathfrak{p}$ since $\mathfrak{p}$ is the common associated prime of $\mathfrak{q}_1$ and $\mathfrak{q}_2$, while $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are not isomorphic since $\mathfrak{q}_1 \neq \mathfrak{q}_2$. Furthermore, it follows from sec. 2 and the fact that $(\mathfrak{q}_1, \mathfrak{q}_2)$ is a common minimal divisor of $\mathfrak{q}_1$ and $\mathfrak{q}_2$, that $\mathfrak{B}_1$ and

$\mathfrak{B}_2$ contain non-zero maximal submodules $\mathfrak{W}_1 \subset \mathfrak{B}_1$ and $\mathfrak{W}_2 \subset \mathfrak{B}_2$ which both have $(\mathfrak{q}_1, \mathfrak{q}_2)$ as annihilator. Hence $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are completely indecomposable A-modules with the same annihilator, which implies, according to Theorem 5.1, that $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are isomorphic. Let J be any fixed isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$ and let $\mathfrak{B}$ be the A-module which is obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. We claim that $\mathfrak{B}$ has the required properties. In the first place, it follows from lemma 8.1 that the annihilating ideal of $\mathfrak{B}$ is $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \mathfrak{q}$. In the second place, since $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are maximal submodules of the not-isomorphic modules $\mathfrak{B}_1$ and $\mathfrak{B}_2$, J is not extendable. Hence all the conditions of corollary 9.3 are satisfied which proves that $\mathfrak{B}$ is completely indecomposable. This completes the proof of Theorem 9.1.

COROLLARY 9.4. *Every completely primary ring A with unit element whose ideals satisfy both chain conditions has a faithful, completely indecomposable representation space.*

Proof. The zero ideal of A is primary and has a maximal associated prime ideal. Hence this zero ideal can be used as the $\mathfrak{q}$ of Theorem 9.1 which proves corollary 9.4.

REMARK 9.2. Let A be a completely primary ring with unit element which is at the same time an algebra of finite rank with respect to a field. Then the dual vector space of the regular representation of A defined in [8], p. 558, coincides with the faithful, completely indecomposable representation space of corollary 9.4. (See example 7.3.) There is no doubt that, if A is not an algebra, the faithful, completely indecomposable representation space of corollary 9.4 must be considered as the correct generalization of the dual vector space of [8]. Now, let A be any commutative ring with unit element whose ideals satisfy both chain conditions. Let $\mathfrak{o}$ be the zero ideal of A and let $\mathfrak{o} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ be the Noether decomposition of $\mathfrak{o}$. The regular representation space of A is the direct sum of the difference modules $A - \mathfrak{q}_1 \dotplus \ldots \dotplus A - \mathfrak{q}_s$; and the ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ satisfy the requirements of Theorem 9.1. Then, if $\mathfrak{B}_1, \ldots, \mathfrak{B}_s$ denote the completely indecomposable A-modules with respectively $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ as annihilators, we define the direct sum $\mathfrak{B}_1 \dotplus \ldots \dotplus \mathfrak{B}_s$ as the dual vector space of A. It follows from lemma II-B of [8], p. 559, that, if A is an algebra of finite rank with respect to a field, this dual vector space coincides with the dual vector space of the regular representation of A as defined in [8]. Again there is no doubt that, if A is not an algebra, the above definition of dual vector space is the correct generalization of the dual vector space of [8].

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University of Southern California

FUNCTIONALS OF BOUNDED FRECHET VARIATION

MARSTON MORSE AND WILLIAM TRANSUE

1. Introduction. In a series of papers which will follow this paper the authors will present a theory of functionals which are bilinear over a product $A \times B$ of two normed vector[1] spaces A and B. This theory will include a representation[2] theory, a variational theory, and a spectral theory. The associated characteristic equations will include as special cases the Jacobi equations of the classical variational theory when $n = 1$, and self-adjoint integro-differential equations of very general type. The bilinear theory is oriented by the needs of non-linear and non-bilinear analysis in the large.

The object of this paper is the proof of several preliminary but fundamental theorems on functionals k with values $k(s, t)$ defined over a rectangle

$$Q \qquad [a \leqq s \leqq a',\ b \leqq t \leqq b'] \tag{1.1}$$

of the (s, t)-plane. We shall require that the Fréchet[3] variation $P(k)$ (defined in 3) be finite, and that for the given a and b,

$$k(a, t) = 0, \qquad k(s, b) = 0 \qquad \{t \in [b, b'],\ s \in [a, a']\}. \tag{1.2}$$

The condition (1.2) does not limit the useful generality of the theorems. The theorems to be proved are the analogues of simple theorems on a functional f with values $f(s)$ defined over an interval $[a, a']$ with a finite total variation $T(f)$.

To state these theorems let f^+ and f^- denote functionals defined by f over $[a, a']$ as follows:

$$\begin{aligned} f^+(s) &= \lim_{u \to 0} f(s + u), && [a < s < a',\ u > 0] \\ f^-(s) &= \lim_{u \to 0} f(s - u), && [a < s < a',\ u > 0] \\ f^+(a) &= f^-(a) = f(a), \\ f^+(a') &= f^-(a') = f(a'). \end{aligned}$$

The theorems to be extended follow:

I. The limits $f(s-)$ and $f(s+)$ exist for $a < s \leqq a'$ and $a \leqq s < a'$ respectively.

II. The points s at which f fails to be continuous are at most countably infinite.

Received June 18, 1948.

[1] See Banach, *Théorie des opérations linéaires* (Warsaw, 1932), chap. IV.

[2] Morse and Transue, "Functionals F Bilinear over the Produce $A \times B$ of Two Pseudo-normed Vector Spaces. I. The Representation of F," *Ann. of Math.* (To be published.)

[3] Fréchet, "Sur les fonctionnelles bilinéaires," *Trans. Amer. Math. Soc.*, vol. 16 (1915), 215-234. In this basic memoir Fréchet obtains a representation of any functional K which is bilinear on $C \times C$, in the form of a repeated Stieltjeas integral with a distribution function k of the above type.

III. The functionals f^+ and f^- are of bounded variation and

$$T(f^+) = T(f^-) \leqq T(f).$$

Fréchet has established [cf. 3] the following for k. For fixed $s \in [a, a']$, the functional on $[b, b']$ whose values are $k(s, t)$ will be denoted by $k(s, \cdot)$. The functional $k(\cdot, t)$ on $[a, a']$ with fixed t is similarly defined. According to Fréchet $T[k(s, \cdot)]$ and $T[k(\cdot, t)]$ are finite with $P(k)$, for fixed s and t respectively. Moreover

$$(1.3) \qquad T[k(s, \cdot)] \leqq P(k), \qquad T[k(\cdot, t)] \leqq P(k).$$

The 2-dimensional extensions of I, II, III will be stated in **2**, and proved in the later sections.

2. The principal theorems. The statement of our theorems requires certain definitions. We shall use the notation f^+, f^- in the sense of **1**. For $s \in [a, a']$ set $k(s, \cdot) = f_s$. For fixed s and $t \in [b, b']$, f_s^+ is well defined. Set

$$f_s^+(t) = k^2(s, t), \qquad f_s^-(t) = k^{-2}(s, t) \qquad [(s, t) \in Q].$$

With $k(\cdot, t) = \phi_t$, for fixed t set

$$\phi_t^+(s) = k^1(s, t), \qquad \phi_t^-(t) = k^{-1}(s, t) \qquad [(s, t) \in Q].$$

Under our hypothesis that $P(k)$ is finite the functionals with values $k^\mu(s, t)$ are well defined over Q for $\mu = \pm 1, \pm 2$. Granting the result of Lemma 3.3 that

$$P(k^\mu) \leqq P(k)$$

and Fréchet's result (1.3), the functionals $(k^\mu)^\nu$ are likewise well defined over Q for $\mu, \nu = \pm 1, \pm 2$. As a matter of notation set $(k^\mu)^\nu = k^{\mu,\nu}$. At inner points (s, t) of Q, for $\mu = 1$, $\nu = 2$ for example,

$$k^{1,2}(s, t) = \lim_{v \to 0+} \left[\lim_{u \to 0+} k(s + u, t + v) \right].$$

In the (u, v)-plane let the four quadrants on which $uv \neq 0$ be designated as follows:

$H_{1,2}$	$[u > 0, v > 0]$,	1st quadrant,
$H_{-1,2}$	$[u < 0, v > 0]$,	2nd quadrant,
$H_{-1,-2}$	$[u < 0, v < 0]$,	3rd quadrant,
$H_{1,-2}$	$[u > 0, v < 0]$,	4th quadrant.

Thus the sign preceding a subscript 1 indicates the sign of u in a quadrant $H_{\mu,\nu}$, while the sign preceding a subscript 2 indicates the sign of v in a quadrant $H_{\mu,\nu}$. It will be convenient to let $H_{\nu,\mu}$ indicate the same quadrant as $H_{\mu,\nu}$ so that

$$H_{\mu,\nu} = H_{\nu,\mu} \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|].$$

With this understood, let (s, t) be an inner point of Q and set

$$(2.1) \qquad \lim_{(u,v) \to (0,0)} k(s + u, t + v) = k^{(\mu,\nu)}(s, t), \qquad [(u, v) \in H_{\mu,\nu}]$$

granting the existence of the limit, as affirmed in Theorem 5.1. This is a limit of k at (s, t) on approaching (s, t) from a specified quadrant with vertex at (s, t). The relation (2.1) defines $k^{(\mu,\nu)}$ at *inner* points of Q. On the edges

of Q, $k^{\mu,\nu}$ is well defined, and we shall complete the definition of $k^{(\mu,\nu)}$ by setting

$$k^{(\mu,\nu)}(s,t) = k^{\mu,\nu}(s,t) \tag{2.2}$$

for (s,t) on any edge of Q.

In order to state the full analogue of I, 1, we need to define an *extension* $\bar{k}$ of k over the whole (s,t)-plane. If L is any line parallel to the coordinate axes and intersecting Q, we require that $\bar{k}$ be constant on the closure of each of the two segments of L exterior to Q. On the closure of each of the four regions in the (s,t)-plane on which $\bar{k}$ is as yet undefined, we require that $\bar{k}$ be constant. These requirements on $\bar{k}$ are consistent as is readily seen.

The generalizations of I, II, and III of 1 can be stated now.

It is assumed throughout that the Fréchet variation $P(k)$ is finite and that k vanishes on the lower and left edges of Q.

THEOREM 2.1. *The limit functionals $k^{\mu,\nu}$ exist at each point of Q. The limits defined by* (2.1) *exist not only at each inner point of Q, but if k is extended over the (s,t)-plane as above, at each point of the (s,t)-plane.*

THEOREM 2.2. *The points of discontinuity of k can be covered by a countable number of lines parallel to the coordinate axes.*

THEOREM 2.3. *On Q, for $\mu, \nu = \pm 1, \pm 2$ with $|\mu| \neq |\nu|$,*

$$k^{(\mu,\nu)}(s,t) = k^{\mu,\nu}(s,t) = k^{\nu,\mu}(s,t),$$

while $P[k^{(\mu,\nu)}(s,t)]$ is independent of admissible μ, ν, and at most $P(k)$.

That $P(k)$ can be finite while the classical total variation $V(k)$ of Vitali is infinite has been shown by example by Clarkson and Adams.[4] The difficult problem which Clarkson and Adams solve corresponds to a general problem which we shall solve in the bilinear theory. In particular it is possible to replace the Clarkson and Adams k with its infinitely many points of discontinuity by a k which is absolutely continuous in each variable separately with $P(k)$ finite and $V(k)$ still infinite. Such an example is essential for the theory of functionals bilinear on $L^p \times L^q$, $p \geqq 1$, $q \geqq 1$.

It is easy to produce an example in which the limit functionals defined by k,

$$k^{1,2},\ k^{-1,2},\ k^{-1,-2},\ k^{1,-2} \tag{2.3}$$

have four different values at some point of Q. One can, for example, suppose that $k(s,t) = 0$ on Q except on some rectangle Q_1 interior to Q. We admit only those rectangles whose edges are parallel to the coordinate axes. Let Q_1 be divided into four congruent rectangles by lines parallel to its edges. On these four rectangles let k have the values 1, 2, 3, 4. The assignment of the dividing lines to the four rectangles is immaterial. At the centre (s_0, t_0) of Q_1 the functionals (2. 3) take on the values 1, 2, 3, 4 in some order.

The fact that the Vitali variation $V(k)$ is in general infinite prohibits the

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[4]Clarkson and Adams, "On Definitions of Bounded Variation for Functions of Two Variables," *Trans. Amer. Math. Soc.*, vol. 35 (1933), 824-854.

use of the Fubini theorem to justify the interchange of the order of integration in the repeated Lebesgue Stieltjes integrals which we shall use in the representation theory. However the equality

$$k^{\alpha,\beta}(s,t) = k^{\beta,\alpha}(s,t) \qquad [(s,t) \in Q]$$

fills the gap, so that the desired interchange of order of integration will be seen to be permissible.

3. Definitions and first properties of $P(k)$. Let $[s, s']$ and $[t, t']$ be subintervals of $[a, a']$ and $[b, b']$ respectively. The "*mixed difference*" determined by k and the rectangle $[s, s'] \times [t, t']$ will be denoted by $[k: s, s': t, t']$. It is given by the equation

$$[k: s, s': t, t'] = k(s', t') - k(s', t) - k(s, t') + k(s, t).$$

Let $s_0, s_1, \ldots, s_n$ and $t_0, t_1, \ldots, t_p$ be values of s and t such that

$$\begin{aligned} a = s_0 &\leqq s_1 \leqq \ldots \leqq s_n = a', \\ b = t_0 &\leqq t_1 \leqq \ldots \leqq t_p = b'. \end{aligned} \tag{3.1}$$

With (3.1) *associate* numbers $e_1, \ldots, e_n$ and $e_1', \ldots, e_p'$ equal to ± 1. For $0 < i \leqq n$, and $0 < j \leqq p$, let

$$\Delta_{ij}(k) = [k : s_i, s_{i-1} : t_j, t_{j-1}]. \tag{3.2}$$

The subdivision of Q defined by the set of lines $s = s_i$ and $t = t_j$ will be called the *partition* π of Q defined by (3.1). Using the summation convention of tensor algebra, let

$$\sup e_i e_j' \Delta_{ij}(k) = P(k), \qquad [i = 1, \ldots, n;\ j = 1, \ldots, p] \tag{3.3}$$

taking the sup over all partitions of Q and associated sets (e) and (e'). We shall term $P(k)$ the *Fréchet variation* of k. We are assuming that $P(k)$ is finite.

The following lemma is due to Fréchet.

LEMMA 3.1. *If $P(k)$ is finite, then for fixed s and t respectively on $[a, a']$ and $[b, b']$*

$$T[k(s, \cdot)] \leqq P(k), \qquad T[k(\cdot, t)] \leqq P(k). \tag{3.4}$$

Using the partition (3.1) set

$$k(s_i, \cdot) - k(s_{i-1}, \cdot) = \Delta_i^1(k), \qquad [i = 1, \ldots, n]$$

and observe[5] that

$$P(k) \geqq \sup T\,[e_i \Delta_i^1(k)], \tag{3.5}$$

where the sup is taken over all partitions of $[a, a']$ and corresponding sets (e) with $e_i = \pm 1$. Since the functional in the bracket [ ] in (3.5) has a null value when $t = b$, it follows that for each t in $[b, b']$, on setting $\Delta_i^1 k(\cdot, t) = k(s_i, t) - k(s_{i-1}, t)$,

$$|e_i \Delta_i^1 k(\cdot, t)| \leqq T[e_i \Delta_i^1(k)] \leqq P(k).$$

Hence the second inequality in (3.4) holds. Interchanging the roles of s and t the first inequality in (3.4) follows similarly.

[5] The equality holds in (3.5).

Lower semi-continuity. (Written l.s.c.) Let G be the class of all functionals which map Q into R_1, the space of real numbers. Convergence of a sequence of elements $g^{(n)} \epsilon G$ to $g \epsilon G$ shall mean pointwise convergence of $g^{(n)}$ to g, that is that

$$\lim_{n \to \infty} g^{(n)}(s, t) = g(s, t) \qquad [(s, t) \epsilon Q].$$

Let R_1, extended by adding the value $+\infty$, be denoted by R_1^+. Let F map G into R_1^+. The value of F at g is denoted by $F(g)$. By definition F is l.s.c. if, whenever $g^{(n)} \to g$ in G,

$$\liminf_{n \to \infty} F[g^{(n)}] \geqq F(g).$$

A sufficient condition that F be l.s.c. is given in the following.

(A). *For each element a in a range (a) let F_a be a l.s.c. map of G into R_1. If for every g in G,*

$$F(g) = \sup_{(a)} F_a(g),$$

then F is l.s.c. over G. Cf. McShane.[6]

LEMMA 3.2. *The Fréchet variation P is l.s.c. over G.*

To prove this lemma we apply (A). To this end let a symbolize any partition (3.1) of Q together with associated sets

$$(e) = (e_1, \ldots, e_{n_a}), \qquad (e') = (e'_1, \ldots, e'_{p_a}).$$

For this partition and sets (e) and (e') set

$$F_a(k) = \left| e_i e'_j \Delta_{ij}(k) \right| \qquad [i = 1, \ldots, n_a;\ j = 1, \ldots, p_a].$$

It is clear that for a fixed a, F_a is l.s.c. over G, in fact continuous. Since

$$P(k) = \sup_{(a)} F_a(k)$$

by definition of $P(k)$, the lemma follows from (A).

LEMMA 3.3. $P(k^\mu) \leqq P(k)$ *for* $\mu = \pm 1, \pm 2$.

Since $P(k)$ is finite the limit functional k^μ exists by Lemma 3.1. To continue we suppose $\mu = 1$.

For each positive integer n, let ϕ_n be a homeomorphic mapping of $[a, a']$ onto $[a, a']$ leaving $s = a$ and $s = a'$ invariant, and such that

$$0 < \phi_n(s) - s < \frac{1}{n} \qquad [a < s < a'].$$

Such a mapping is seen to exist. In terms of k set

$$k[\phi_n(s), t] = k_n(s, t) \qquad [n = 1, 2, \ldots]. \tag{3.6}$$

It is clear that

$$\lim_{n \to \infty} k_n(s, t) = k^1(s, t). \tag{3.7}$$

For each n however

$$P(k) = P(k_n). \tag{3.8}$$

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[6] McShane, *Integration* (Princeton, 1944), 41.

To establish (3.8) let π be a partition of Q given by (3.1) with (e) and (e') corresponding sets and Δ_{ij}^{π} the corresponding mixed difference operator. Let π_n be a partition in which s_i in (3.1) is replaced by $\phi_n(s_i)$ and t_j remains unchanged, while $\Delta_{ij}^{\pi_n}$ is the corresponding mixed difference operator. With k_n defined by (3.6) it is clear that

$$e_i e'_j \Delta_{ij}^{\pi}(k) = e_i e'_j \Delta_{ij}^{\pi_n}(k_n).$$

Relation (3.8) follows.

Since $k_n \to k^1$ pointwise, and $P(g)$ is l.s.c.

$$\liminf_{n\to\infty} P(k_n) \geqq P(k^1).$$

From (3.8) therefore, $P(k) \geqq P(k^1)$.

The cases in which $\mu = -1$, or ± 2 are similar.

THEOREM 3.1. *The limit functionals $k^{\mu,\nu}$ exist for $\mu, \nu = \pm 1, \pm 2$, and $P(k^{\mu,\nu}) \leqq P(k)$.*

When $P(k)$ is finite k^{μ} exists by Lemma 3.1, and $P(k^{\mu}) \leqq P(k)$ by Lemma 3.3. Hence by Lemma 3.1, $(k^{\mu})^{\nu} = k^{\mu,\nu}$ exists, and by Lemma 3.3

$$P(k^{\mu,\nu}) \leqq P(k).$$

A second definition of $P(k)$. Corresponding to the partition (3.1) of Q, let

$$(\eta_1, \ldots, \eta_n), \quad (\eta'_1, \ldots, \eta'_p)$$

be sets of constants at most 1 in absolute value, in particular admitting the value 0. Taking the sup over all partitions (3.1) of Q and corresponding sets (η) and (η'), set

$$P'(k) = \sup \eta_i \eta'_j \Delta_{ij}(k) \quad [i = 1, \ldots, n;\ j = 1, \ldots, p]. \tag{3.9}$$

LEMMA 3.4. $P(k) = P'(k)$.

It is immediately clear that $P(k) \leqq P'(k)$. It remains to show that $P'(k) \leqq P(k)$. To that end set

$$e_i = \text{sign } [\eta'_j \Delta_{ij}(k)].$$

It then appears that for a partition (3.1) of Q

$$\eta_i \eta'_j \Delta_{ij}(k) \leqq e_i \eta'_j \Delta_{ij}(k).$$

Similarly if

$$e'_j = \text{sign } [e_i \Delta_{ij}(k)],$$

$$e_i \eta'_j \Delta_{ij}(k) \leqq e_i e'_j \Delta_{ij}(k) \leqq P(k)$$

from which it follows that $P'(k) \leqq P(k)$.

4. A fundamental lemma. Let (σ, τ) be a point interior to Q. Let

$$\begin{aligned} a < s_1 < s'_1 < s_2 < s'_2 < \ldots \\ b < t_1 < t'_1 < t_2 < t'_2 < \ldots \end{aligned} \tag{4.1}$$

be increasing sequences of values of s and t respectively converging to σ and τ. Let r and m be arbitrary positive integers. Denote the mixed difference determined by k and the rectangle $[s_r, s'_r] \times [t_m, t'_m]$ by $[k: r, m]$.

Let δ be a positive constant. With $[s_r, s_r']$ fixed, for $\tau - t_m$ sufficiently small and for $t_m < t_m' < \tau$,

$$[k: r, m] < \delta. \tag{4.2}$$

This follows from the existence of the limits

$$k(s_r, \tau-), \quad k(s_r', \tau-),$$

recalling that these limits exist since $P(k)$ is finite. Similarly, for fixed $[t_m, t_m']$, for $\sigma - s_r$ sufficiently small, and for $s_r < s_r' < \sigma$, (4.2) again holds.

Let (s, t) and (s', t') be points of Q for which

$$a < s < s' < \sigma, \quad b < t < t' < \tau, \tag{4.3}$$

and as previously let $[k: s, s': t, t']$ be the mixed difference determined by k and the rectangle $[s, s'] \times [t, t']$. The following lemma is fundamental.

LEMMA 4.1. *Lim* $[k: s, s': t, t'] = 0$ *as* $(s, t) \to (\sigma, \tau)$ *with* (4.3.) *holding.*

Given $e > 0$, the lemma affirms in effect that for $\sigma - s$ and $\tau - t$ sufficiently small and positive, and with (4.3) holding

$$\left|[k: s, s': t, t']\right| \leqq e. \tag{4.4}$$

Let e_{rm} be a double sequence of positive constants such that

$$\Sigma\, e_{rm} < e \qquad [r, m = 1, 2, \ldots].$$

Assume the lemma false. Then for some $e > 0$, (4.4) will fail to hold subject to (4.3), for $\sigma - s$ and $\tau - t$ sufficiently small. One can accordingly choose the numbers

$$[s_m, s_m', t_m, t_m'] \tag{4.5}$$

in the order of increasing m so that (4.1) holds, so that $s_m \to \sigma$, $t_m \to \tau$, and so that the following conditions are successively satisfied. For $m = 1$, the set (4.5) is to be such that

$$\left|[k: 1, 1]\right| > e.$$

For $m = 2$, the set (4.5) is to be such that the order (4.1) holds and

$$\left|[k: 2, 2]\right| > e; \quad \left|[k: 1, 2]\right| < e_{12}; \quad \left|[k: 2, 1]\right| < e_{21}. \tag{4.6}$$

The last two conditions will be fulfilled if $\sigma - s_2$ and $\tau - t_2$ are sufficiently small and positive, as has been remarked in connection with (4.2). For a general m the set (4.5) is to be such that (4.1) holds,

$$\left|[k: m, m]\right| > e,$$

and for $r = 1, 2, \ldots, m - 1$,

$$\left|[k: r, m]\right| < e_{rm}, \quad \left|[k: m, r]\right| < e_{mr}.$$

A partition $\pi^{(m)}$ of Q will be defined now by the points of division

$$a = s_0 < s_1 < s_1' < s_2 < s_2' < \ldots < s_m < s_m' < a', \tag{4.7}$$

$$b = t_0 < t_1 < t_1' < t_2 < t_2' < \ldots < t_m < t_m' < b'. \tag{4.8}$$

To the $2m + 1$ subintervals into which $[a, a']$ and $[b, b']$ are thereby divided, constants

$$[\eta_1, \ldots, \eta_{2m+1}], \quad [\eta_1', \ldots, \eta_{2m+1}']$$

with $|\eta_i| \leqq 1$, $|\eta_j'| \leqq 1$, will be respectively assigned. In accordance with the definition (3.9) of $P'(k) = P(k)$,

$$P(k) \geqq \eta_i \eta_j' \Delta_{ij}(k) \qquad [i, j = 1, 2, \ldots, 2m+1]$$

for the partition $\pi^{(m)}$ and sets (η) and (η'). We set

$$(\eta) = [0, \text{sign } [k: 1, 1], 0, \text{sign } [k: 2, 2], \ldots, \text{sign } [k: m, m], 0],$$
$$(\eta') = [0, 1, 0, 1, \ldots, 1, 0].$$

For the partition $\pi^{(m)}$, and above choice of (η) and (η')

$$\eta_i \eta_j' \Delta_{ij}(k) = \Sigma \eta_i \eta_i' \Delta_{ii}(k) + \sum_{i \neq j} \eta_i \eta_j' \Delta_{ij}(k)$$
$$= \sum_{r=1}^{m} |[k: r, r]| + R,$$

where

$$|R| < \Sigma e_{rp} < e, \qquad [r, p = 1, \ldots, m]$$

so that

$$\eta_i \eta_j' \Delta_{ij}(k) > me - e.$$

We conclude that $P(k) > (m-1)e$. Since m is an arbitrary integer $P(k) = \infty$. From this contradiction we infer the truth of the lemma.

5. The limit functions $k^{(\mu,\nu)}$. Given values of s and t

$$a < s_1 < s_2 < s_3 < a',$$
$$b < t_1 < t_2 < t_3 < b', \tag{5.1}$$

we shall establish the formula

$$\begin{aligned}[k: a, s_3: b, t_3] &= [k: a, s_3: b, t_3] + [k: a, s_1: t_2, t_3] \\ &+ [k: s_2, s_3: b, t_1] + [k: s_1, s_3: t_1, t_3] \\ &- [k: s_1, s_2: t_1, t_2].\end{aligned} \tag{5.2}$$

This formula results from the fact that mixed differences are additive functions of intervals (2-dimensional).

To establish (5.2) we first represent the rectangle $[a, s_3] \times [b, t_3]$ as the union of four non-overlapping rectangles,

$$\begin{aligned}[a, s_3] \times [b, t_3] &= [a, s_1] \times [b, t_1] + [a, s_1] \times [t_1, t_3] \\ &+ [s_1, s_3] \times [b, t_1] + [s_1, s_3] \times [t_1, t_3],\end{aligned}$$

corresponding to which one has the relation,

$$\begin{aligned}[k: a, s_3: b, t_3] &= [k: a, s_1: b, t_1] + [k: a, s_1: t_1, t_3] \\ &= [k: s_1, s_3: b, t_1] + [k: s_1, s_3: t_1, t_3].\end{aligned} \tag{5.3}$$

A decomposition of $[a, s_3] \times [b, t_3]$ gives the relation

$$\begin{aligned}[k: a, s_3: b, t_3] &= [k: a, s_1: b, t_1] + [k: a, s_1: t_1, t_3] \\ &+ [k: a, s_1: t_2, t_3] + [k: s_1, s_3: b, t_1] \\ &+ [k: s_2, s_3: b, t_1] + [k: s_1, s_3: t_1, t_3].\end{aligned} \tag{5.4}$$

If one adds $[k: s_1, s_2: t_1, t_2]$ to both sides of (5.4), and makes use of (5.3) in simplifying the resulting right member, one has

$$\begin{aligned}&[k: a, s_2: b, t_2] + [k: s_1, s_2: t_1, t_2]\\ &= [k: a, s_2: b, t_2] + [k: a, s_1: t_2, t_2]\\ &+ [k: s_2, s_2: b, t_1] + [k: s_1, s_2: t_1, t_2].\end{aligned} \tag{5.5}$$

Relation (5.5) differs from (5.2) only in the transposition of one term.

LEMMA 5.1. *At each interior point* (σ, τ) *of* Q

$$\lim_{(u,v)\to(0,0)} k(\sigma - u, \tau - v) \qquad [u > 0, v > 0]$$

exists.

To prove the lemma the values appearing in (5.1) will be taken so that

$$\begin{aligned}a < s_1 < s_2 < s_2 < \sigma,\\ b < t_1 < t_2 < t_2 < \tau.\end{aligned} \tag{5.6}$$

Let e be a positive constant. Referring to (5.2), choose s_1 and t_1 respectively so near σ and τ, say

$$0 < \sigma - s_1 < \delta_1, \quad 0 < \tau - t_1 < \delta_1,$$

that

$$\left|[k: s_1, s_2: t_1, t_2]\right| < e, \quad \left|[k: s_1, s_2: t_1, t_2]\right| < e. \tag{5.7}$$

This is possible by virtue of Lemma 4.1. With (s_1, t_1) so chosen and held fast, choose s_2 and t_2 respectively so near σ and τ, say

$$0 < \sigma - s_2 < \delta_2, \quad 0 < \tau - t_2 < \delta_2,$$

that

$$\left|[k: a, s_1: t_2, t_2]\right| < e, \quad \left|[k: s_2, s_2: b, t_1]\right| < e. \tag{5.8}$$

This is possible because of the existence of the limits $k(s, \tau-)$ and $k(\sigma-, t)$ respectively. With these choices of s_1, s_2, t_1, t_2 it follows from (5.2) that

$$\left|[k: a, s_2: b, t_2] - [k: a, s_2: b, t_2]\right| < 4e. \tag{5.9}$$

Recall that $[k: a, s: b, t] = k(s, t)$ since k vanishes on the lower and left edges of Q. It follows from (5.9) that if

$$\begin{aligned}0 < \sigma - s < \delta_2, \quad 0 < \sigma - s' < \delta_2,\\ 0 < \tau - t < \delta_2, \quad 0 < \tau - t' < \delta_2,\end{aligned}$$

then

$$\left|k(s, t) - k(s', t')\right| < 8e. \tag{5.10}$$

The lemma is a ready consequence of (5.10).

LEMMA 5.2. *At each interior point* (σ, τ) *of* Q

$$\lim_{(u,v)\to(0,0)} k(\sigma + u, \tau + v) \qquad [u > 0, v > 0]$$

exists.

Lemma 5.2 can be deduced from Lemma 5.1. To that end make a transformation

$$s - a = a' - s', \quad t - b = b' - t' \tag{5.11}$$

of Q onto Q, interchanging the upper and lower, and the right and left edges of Q. Under (5.11) set

$$h(s, t) = h(s', t').$$

We have $P(k) = P(h)$, but cannot apply Lemma 5.1 to h since $h(a, t')$ and $h(s', b)$ are not necessarily zero for $t' \in [b, b']$ and $s' \in [a, a']$. We accordingly introduce the functional g with values

$$g(s', t') = h(s', t') - h(a, t') - h(s', b) + h(a, b),$$

for which $g(a, t') = g(s', b) = 0$ as desired. Observe also that $P(g) = P(h)$.

If then (σ', τ') is any interior point of Q, it follows from the preceding lemma that the

$$\lim_{(u,v)\to(0,0)} g(\sigma' - u, \tau' - v) \qquad [u > 0, v > 0] \tag{5.12}$$

exists. Since the limits $h(a, \tau-)$ and $h(\sigma-, b)$ exist, it follows from (5.12) that a limit similar to (5.12) exists for h. But for the image (σ, τ) of (σ', τ') under (5.11),

$$\lim_{(u,v)\to(0,0)} h(\sigma' - u, \tau' - v) = \lim_{(u,v)\to(0,0)} k(\sigma + u, \tau + v) \qquad [u > 0, v > 0]$$

and the proof of the lemma is complete.

On using the transformation

$$s - a = a' - s', \quad t = t',$$

the existence of the limit

$$\lim_{(u,v)\to(0,0)} k(\sigma + u, \tau - v) \qquad [u > 0, v > 0]$$

is similarly deduced from Lemma 5.1. The transformation

$$s = s', \qquad t - b = b' - t',$$

is similarly used to establish the limit

$$\lim_{(u,v)\to(0,0)} k(\sigma - u, \tau + v) \qquad [u > 0, v > 0].$$

The limits defined in (2.1) have thus been proved to exist at each interior point of Q.

If moreover k is given an extension $\bar{k}$ over the whole (s, t)-plane as in **2**, the limits defined in (2.1) exist at each point of the plane. To see this, observe that the mixed difference determined by $\bar{k}$ and any admissible rectangle X (with edges parallel to the coordinate axes) vanishes when the interior of X is on the complement of Q. Hence if Q' is any admissible rectangle which contains Q in its interior, the Fréchet variation of $\bar{k}$ over Q' equals $P(k)$ taken over Q. The edges of Q are interior to Q' so that the limits (2.1) taken for $\bar{k}$ exist at points on these edges. We thus have the theorem.

THEOREM 5.1. *The limits* (2.1), *evaluated for the extension* $\bar{k}$ *of* k *over the* (s, t)*-plane, exist at each point of the* (s, t)*-plane.*

6. The derived theorems. We begin with the lemma.

LEMMA 6.1. *At each interior point* (s, t) *of* Q

$$k^{(\mu,\nu)}(s,t) = k^{\mu,\nu}(s,t) \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|]. \tag{6.0}$$

The proof for the case $\mu = 1, \nu = 2$ is typical. We begin with the inequality

$$\begin{aligned} |k^{(1,2)}(s,t) - k^{1,2}(s,t)| \leqq\ & |k^{(1,2)}(s,t) - k(s+u, t+v)| \\ &+ |k(s+u, t+v) - k^{1,2}(s,t+v)| \\ &+ |k^{1,2}(s,t+v) - k^{1,2}(s,t)|. \end{aligned} \qquad [u > 0, v > 0] \tag{6.1}$$

Let e be a positive constant. Since $k(s+u, t+v) \to k^{(1,2)}(s,t)$ as $u \to 0+$, $v \to 0+$, there exists a $\delta < 0$ such that the first term on the right of (6.1) is at most e, provided

$$0 < u < \delta, \quad 0 < v < \delta.$$

Since $k^{1,2}(s, t+v) \to k^{1,2}(s,t)$ as $v \to 0+$, there exists a δ_2 with $0 < \delta_2 < \delta$, such that when $0 < v < \delta_2$ the third term on the right of (6.1) is at most e. Fix v with $0 < v < \delta_2$; there will then exist a δ_1 with $0 < \delta_1 < \delta$ such that the second term on the right is at most e when $0 < u < \delta_1$. For this choice of v and u, (6.1) shows that

$$|k^{(1,2)}(s,t) - k^{1,2}(s,t)| < 3e. \tag{6.2}$$

Since the left member of (6.2) is independent of (u, v)

$$k^{(1,2)}(s,t) = k^{1,2}(s,t).$$

The lemma follows similarly for other values of μ and ν.

Since the relation (6.0) holds on the edges of Q by definition of $k^{(\mu,\nu)}$ in (2.2), the following theorem is established.

THEOREM 6.1. *At each point of* Q

$$k^{(\mu,\nu)}(s,t) = k^{\mu,\nu}(s,t) \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|]. \tag{6.3}$$

LEMMA 6.2. *On the edges of* Q

$$k^{\mu,\nu}(s,t) = k^{\nu,\mu}(s,t) \qquad |\mu| \neq |\nu|. \tag{6.4}$$

The existence of the limit functions follows from Lemmas 3.1 and 3.3.

For points on the lower and left edges of Q

$$0 = k(s,t) = k^{\mu}(s,t) = k^{\nu}(s,t) = k^{\mu,\nu}(s,t) = k^{\nu,\mu}(s,t).$$

On the right and upper edges of Q respectively

$$\begin{aligned} k^{\mu}(a',t) &= k(a',t), & |\mu| &= 1, \\ k^{\mu}(s,b') &= k(s,b'), & |\mu| &= 2, \end{aligned}$$

by definition of k^{μ} in 2. Applying these relations one finds for $|\mu| \neq |\nu|$, that

$$k^{\mu,\nu}(a',t) = k^{\nu}(a',t), \qquad |\nu| = 2,$$

since $|\mu|$ is then 1, and $k^{\mu}(a',t) = k(a',t)$.

Similarly

$$\begin{aligned} k^{\mu,\nu}(a',t) &= k^{\mu}(a',t), & |\mu| &= 2, \\ k^{\mu,\nu}(s,b') &= k^{\nu}(s,b'), & |\nu| &= 1, \\ k^{\mu,\nu}(s,b') &= k^{\mu}(s,b'), & |\mu| &= 1. \end{aligned}$$

The last four relations establish (6.4) on the right and upper edges of Q.

THEOREM 6.2. *At each point* (s, t) *of* Q

$$k^{\mu,\nu}(s,t) = k^{\nu,\mu}(s,t) \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|]. \tag{6.5}$$

At each interior point (s, t) of Q

$$k^{(\mu,\nu)}(s,t) = k^{(\nu,\mu)}(s,t)$$

by definition of $k^{(\mu,\nu)}$ and $k^{(\nu,\mu)}$ in 2. At such a point of Q, (6.5) accordingly follows from (6.3). On the edges of Q, (6.5) follows from Lemma 6.2.

THEOREM 6.3. *The points at which* k *fails to be continuous can be covered by a countable set of lines parallel to the coordinate axes.*

Given $e > 0$, let S_e be a set of points p of Q with the following properties. At p, k shall be discontinuous with a jump exceeding e, and no two points of S_e shall have the same s or t coordinate. The maximum number N_e of points in such sets S_e is finite. Otherwise there would exist a set S_e containing an infinite set of points p, and these points would have some point (σ, τ) of Q as a limit point. In one at least of the four open quadrants defined by the lines $s = \sigma$, $t = \tau$, there would be an infinite subset of points of S_e. This is impossible if the limits $k^{(\mu,\nu)}(\sigma, \tau)$ exist. Hence N_e is finite.

A countable set Ω of lines parallel to the coordinate axes on which all points of discontinuity of k lie can be enumerated as follows.

With $e = 1$, let S_1 be a set of points of Q defined as above with S_1 maximal in that S_1 contains just N_1 points. In Ω take first the lines $s = s_1$, $t = t_1$ determined by points (s_1, t_1) of S_1. Let $S_{.1}$ be a second maximal set. To Ω now add all lines $s = s_2$, $t = t_2$ corresponding to points of $S_{.1}$ at which the jump of k does not exceed 1. Let $S_{.01}$ be a third maximal set, and add to Ω all lines $s = s_3$, $t = t_3$ corresponding to points (s_3, t_3) of $S_{.01}$ at which the jump of k does not exceed .1, etc. The resulting set of lines is at most countable and will cover the points of discontinuity of k.

THEOREM 6.4. *The value of*

$$P[k^{(\mu,\nu)}] \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|] \tag{6.6}$$

is independent of μ *and* ν, *and at most* $P(k)$.

We begin by proving Lemma 6.3.

LEMMA 6.3. *Let* (μ, ν) *be a pair admitted in* (6.6) *and* (μ', ν') *a second such pair. On setting* $k^{(\mu',\nu')} = h$ *one has the relation*

$$h^{(\mu,\nu)} = k^{(\mu,\nu)}. \tag{6.7}$$

In accordance with the preceding theorem the points at which h and k are continuous and equal are everywhere dense on G. Hence (6.7) holds at each interior point of G. On the lower and left edges of G both members of (6.7) vanish and so are equal. The proof that (6.7) holds on the edge of G on which $s = a'$ is as follows.

Of the members μ, ν one, say σ, is ± 2, and of the members μ', ν' one, say τ, is ± 2. In accordance with the equalities used in proving Lemma 6.2,

$$k^{\mu',\nu'}(a',t) = k^{\tau}(a',t) \quad h^{\mu,\nu}(a',t) = h^{\sigma}(a',t).$$

Using these relations and Lemma 6.1

$$h^{\mu,\nu}(a',t) = h^{\nu}(a',t) = [k^{\mu',\nu'}(a',t)]^{\nu}$$
$$= [k^{\nu}(a',t)]^{\nu} = k^{\nu}(a',t) = h^{\mu,\nu}(a',t),$$

from which (6.7) follows on the right edge of Q. The validity of (6.7) on the upper edge of Q is similarly proved.

Proof of Theorem 6.4. Set

$$k^{(\mu',\nu')} = h \qquad k^{(\mu,\nu)} = g.$$

In accordance with (6.7), $h^{(\mu,\nu)} = g$ so that it follows from Theorem 6.1 and Theorem 3.1 that

$$P(h) \leqq P(g) \leqq P(k).$$

Similarly $g^{(\mu',\nu')} = h$ so that

$$P(g) \leqq P(h) \leqq P(k).$$

Hence

$$P(h) = P(g) \leqq P(k).$$

This completes the proof of the theorem.

The Institute for Advanced Study
Kenyon College

ON THE DISJOINT PRODUCT OF IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

G. DE B. ROBINSON

1. Introduction. The results of the present paper can be interpreted (*a*) in terms of the theory of the representations of the *symmetric group*, or (*b*) in terms of the corresponding theory of the *full linear group*. In the latter connection they give a solution to the problem of the expression of an invariant matrix of an invariant matrix as a sum of invariant matrices, in the sense of Schur's Dissertation. D. E. Littlewood[1] has pointed out the significance of this problem for invariant theory and has attacked it via Schur functions, i.e. characters of the irreducible representations of the full linear group. We shall confine our attention here to the interpretation (*a*). Our results are explicit and yield the interpretation (*b*) merely by changing[2] brackets [ ] and multiplication $(\cdot)$ into brackets $\{\ \}$ and multiplication $(\times)$; moreover, they are self-contained from a group-theoretic point of view.

Consider the symmetric group S_{mn} on mn symbols and the subgroup

$$H = S_m \times S_m \times \ldots \times S_m, \qquad n \text{ factors}, \tag{1.1}$$

which is the direct product of n factors S_m on n different sets of the mn symbols. An irreducible representation of H is necessarily the Kronecker product of irreducible representations of the n factors; such an irreducible representation of H *induces* a representation of S_{mn} which is reducible and of degree[3]

$$\frac{(mn)!}{(m!)^n}\, x_\alpha x_\beta \ldots x_\gamma, \tag{1.2}$$

where $x_\alpha, x_\beta, \ldots, x_\gamma$ are the degrees of the n irreducible representations $[\alpha]$, $[\beta], \ldots, [\gamma]$ of the S_m in 1.1. We shall assume in the sequel that such irreducible representations are always written in Young's orthogonal form.

Murnaghan[4] studied these representations of S_{mn} and gave a method for finding their irreducible components. He called such an induced representation the *direct product* of the n irreducible representations $[\alpha], [\beta], \ldots, [\gamma]$. In a previous paper the author has shown[3] that this induced representation is associated with a skew Young diagram which has the n right diagrams $[\alpha]$, $[\beta], \ldots, [\gamma]$ as its disjoint constituents; the skew diagram may be denoted

$$[\alpha] \,.\, [\beta] \,.\, \ldots \,.\, [\gamma], \qquad n \text{ factors}, \tag{1.3}$$

Received May 22, 1948.

[1] [2] and [3]. Cf. also [13] p. 110.

[2] [8] section 3.

[3] [7] and [8].

[4] [5].

where the order of writing the factors is immaterial. This suggests the term *disjoint product*, which designation will be used throughout this paper.

Having associated the induced representation under consideration with a skew Young diagram we have available a second method for finding the irreducible components, namely that based on the Littlewood-Richardson rule.[5] We may assume, then, that these irreducible components are known in any given case.

The problem to be considered here is a certain reduction of the disjoint product in the case where *all the disjoint constituents are the same*; 1.3 becomes

$$1.4 \qquad \Pi_n^{[a]} = [a] \,.\, [a] \,.\, \ldots \,.\, [a], \qquad n \text{ factors},$$

and the reduction in question arises from the interchangeability of the n factors. As one might expect, there is a very close analogy here with Schur's celebrated derivation of the irreducible representations of order n of the full linear group.[6]

The notation $\{a\}\otimes\{\beta\}$ was introduced by Littlewood to designate his "new multiplication" of Schur functions.[7] Restricting our attention to the symmetric group, we obtain the desired reduction of $\Pi_n^{[a]}$ in the following form:

$$1.5 \qquad \Pi_n^{[a]} = \sum_{\beta} x_\beta [a]\odot[\beta],$$

where β runs over all partitions of n, and the reducible component $[a]\odot[\beta]$, which corresponds to Littlewood's $\{a\}\otimes\{\beta\}$, appears with a frequency x_β. The degree of $[a]\odot[\beta]$ is

$$1.6 \qquad \frac{(mn)!}{(m!)^n n!}\,(x_a)^n x_\beta.$$

The frequency $\mathbf{f}$ with which a given irreducible representation $[\lambda]$ of S_{mn} appears as a component of $[a]\odot[\beta]$ is obtained in **5** below and this leads to an immediate proof of Littlewood's Theorem of Conjugates.[8] Littlewood has given a number of methods for obtaining the reduction of $\{a\}\otimes\{\beta\}$ or analogously of $[a]\odot[\beta]$, but none lead to explicit results in the general case. The application of the method developed here is illustrated in **7** below.

Two additional remarks should be made: (*i*) The reason for not applying Schur's original procedure to the Kronecker product $\{a\}^n$ is that the interpretation of the results must ultimately be based, as in Schur's case, upon the theory of the symmetric group. Basing the whole argument on this theory is of some intrinsic interest. (*ii*) While the results of this paper do lead to a systematic study of the structure of the representation $[a]\odot[\beta]$, they do not provide a means of identifying the particular $[a]\odot[\beta]$ to which a given irreducible component $[\lambda]$ of $\Pi_n^{[a]}$ belongs. This identification is important in

[5] Applied to skew diagrams in [7].

[6] [11].

[7] In particular [3] Part II. Note that we use the symbol $\{a\}$ to designate an irreducible representation of the full linear group and not merely its character or corresponding Schur function.

[8] [3] p. 359.

the application to invariant theory.[1] However, the definition of $[\alpha]\odot[\beta]$ as an induced representation in 6 points the way in which this refinement of the theory may be sought.

In concluding this introduction I should like to express my thanks to R. Steinberg and W. T. Sharp for reading the manuscript and suggesting, in particular, the insertion of section 6.

2. The normalizer of H. The subgroup H, as defined in 1.1, has a normalizer in S_{mn} which can be written

$$\mathfrak{N}(H) = H + Hs^*_2 + \ldots + Hs^*_n. \tag{2.1}$$

If each of the n sets of m symbols be considered as an *ordered* set, then the element s^*_i of S_{mn} permutes the sets amongst themselves, preserving this order. The factor group $\mathfrak{N}(H)/H$ satisfies the following relation

$$\mathfrak{N}(H)/H \sim S^*_n \sim S_n, \tag{2.2}$$

and the s^*_i generate the subgroup S^*_n of $\mathfrak{N}(H)$. The normalizer $\mathfrak{N}(H)$ is not a direct product but its structure is sufficiently simple for us to obtain the necessary information concerning it.

With regard to the isomorphism between S_n and S^*_n, let us assume that an element s_i of S_n has ω_λ cycles of length λ so that

$$n = \omega_1 + 2\omega_2 + 3\omega_3 + \ldots + n\omega_n; \tag{2.3}$$

then the corresponding element s^*_i of S^*_n has $m\omega_\lambda$ cycles of length λ.

Given an irreducible representation

$$[\alpha]^n = [\alpha] \times [\alpha] \times \ldots \times [\alpha], \qquad n \text{ factors}, \tag{2.4}$$

of H one naturally asks whether it can be extended to yield a representation of $\mathfrak{N}(H)$, of the same degree. Taking the x_α standard diagrams as variables of $[\alpha]$ and forming all combinations of n of these diagrams, one diagram being associated with each of the n sets of m symbols, we obtain the variables of the Kronecker product $[\alpha]^n$. *The matrices s_i which must be adjoined to those of $[\alpha]^n$ to yield a representation of $\mathfrak{N}(H)$ are just those permutation matrices which arise through transformation by s^*_i of the $(x_\alpha)^n$ variables of $[\alpha]^n$.* We shall say that this representation of $\mathfrak{N}(H)$, which is necessarily irreducible, is *generated*[2] by the representation $[\alpha]^n$ of H.

It will be important in the sequel to have specific information concerning these permutation matrices. To this end let us consider those elements of H which commute with a given s^*_i; clearly they will form a subgroup of H which we call the *centralizer* of s^*_i in H, denoting it $\mathfrak{C}(s^*_i)$. In particular, if $s^*_i = I$ then $\mathfrak{C}(I) = H$. On the other hand, if s^*_i contains m cycles of length n then $\mathfrak{C}(s^*_i) \sim S_m$; each element of $\mathfrak{C}(s^*_i)$ contains n cycles of equal length, in which case we shall say that the n sets of symbols are *linked*. In general, to each of the ω_λ cycles of length λ of s_i there corresponds a factor of $\mathfrak{C}(s^*_i)$ which is

[2] For a further discussion of the irreducible representations of $\mathfrak{N}(H)$ see 6 below.

isomorphic to S_m and in which λ sets of symbols are linked. Thus

2.5 $$\mathfrak{G}(s_i) \sim S_m \times S_m \times \ldots \times S_m, \qquad \omega \text{ factors},$$

where $0 < \omega = \omega_1 + \omega_2 + \ldots + \omega_n \leqslant n$.

We can now prove the following theorem:

2.6 *The characteristic of the permutation matrix $\bar{s}_i$ representing s^*_i in the representation of $\mathfrak{N}(H)$ generated by the representation $[a]^n$ of H is $(x_a)^\omega$ where ω is the number of cycles in s_i.*

To see this it is only necessary to look for the number of variables of $[a]^n$ which remain unaltered after transformation by s^*_i. But this is precisely x_a for each factor S_m in 2.5. Since we are dealing with a Kronecker product the total number of such variables is $(x_a)^\omega$.

Of course if $[a] = [m]$ this whole analysis falls to the ground for then $x_a = 1$ and the representation of $\mathfrak{N}(H)$ under consideration is merely the identity representation.

To illustrate these ideas consider the case $m = 3$, $n = 2$ with $[a] = [2, 1]$. We take the two sets of symbols to be 1, 2, 3; 4, 5, 6, so that

2.7 $$H = \{1 + (12) + (13) + (23) + (123) + (132)\} \times \{1 + (45) + (46) + (56) + (456) + (465)\},$$

2.8 $$\mathfrak{N}(H) = H + H(14)(25)(36),$$

2.9 $$S^*_2 = 1 + (14)(25)(36) \sim S_2.$$

If $s^* = (14)(25)(36)$, then clearly

2.10 $$\mathfrak{G}(s^*) = 1 + (12)(45) + (13)(46) + (23)(56) + (123)(456) + (132)(465).$$

In order to construct the matrix $\bar{s}$ which represents s^* in the representation of $\mathfrak{N}(H)$ generated by $[2, 1]^2$, let us denote the variables of $[2, 1]^2$ by the products of the corresponding standard Young diagrams:

2.11 $$\begin{pmatrix}12\\3\end{pmatrix}\begin{pmatrix}45\\6\end{pmatrix}, \quad \begin{pmatrix}13\\2\end{pmatrix}\begin{pmatrix}45\\6\end{pmatrix}, \quad \begin{pmatrix}12\\3\end{pmatrix}\begin{pmatrix}46\\5\end{pmatrix}, \quad \begin{pmatrix}13\\2\end{pmatrix}\begin{pmatrix}46\\5\end{pmatrix},$$

in this order. Transforming by s^* we have

2.12 $$\bar{s} : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. The reduction of $\Pi_n^{[a]}$. We are interested in the representation of S_{mn} induced[16] by the representation $[a]^n$ of H. If

3.1 $$S_{mn} = \mathfrak{N}(H) + \mathfrak{N}(H)R_2 + \ldots + \mathfrak{N}(H)R_d,$$

where $d = (mn)!/(m!)^n n!$, we can set $s^*_i = s_i$ with no ambiguity for the time

[16] Cf. [12] pp. 198-202.

being and write the group matrix of the induced representation in question in the form

$$3.2 \qquad \Pi_I(H) = \Pi_n^{[a]}(H) = [r_p^{-1} s_i^{-1} \mathbf{H} s_j r_q]$$

where $i, j = 1, 2, \ldots, n!$; $p, q = 1, 2, \ldots, d$ and $\mathbf{H}$ is the group matrix of the representation $[a]^n$ of H. The elements s, r to the right and left of $\mathbf{H}$ operate on the *labels* attached to the independent variables of $\mathbf{H}$ to yield the independent variables of the group matrix of S_{mn}.

Following Schur[6] we seek matrices which will commute with $\Pi_I(H)$. Now it is well known[11] that every matrix of the *left* regular representation $(g_i \to g^{-1} g_i)$ commutes with every matrix of the *right* regular representation $(g_i \to g_i\, g)$ of a given group G. The form 3.2 of the group matrix of S_{mn} is derived from the *right* regular representation of $S_n \sim \mathfrak{N}(H)/H$. Let us construct the matrix

$$3.3 \qquad \Pi_s(I) = (s_j s_k^{-1}) \times I_{(a_k)^n} \times I_d,$$

where $(s_j s_k^{-1})$ is the matrix of the *left* regular representation of S_n representing $s = s_j s_k^{-1}$ and the I's are unit matrices of indicated degrees. By definition,

$$3.4 \qquad \Pi_s(H) = \Pi_s(I) \,.\, \Pi_I(H) = \Pi_I(H) \,.\, \Pi_s(I)$$

where the matrix $I_{(a_k)^n}$ in $\Pi_s(I)$ is placed in the position of $\mathbf{H}$ so that $\Pi_s(I)$ commutes with $\Pi_I(H)$. Multiplying the matrices in 3.4, we obtain a contribution only when $s_j = s s_k$ so that

$$3.5 \qquad \Pi_s(H) = [r_p^{-1} s_i^{-1} s\, \mathbf{H}\, s_k r_q] = [r_p^{-1} s_i^{-1}\, \mathbf{H}\, s s_k r_q] \,.$$

These two apparently different designations of the independent variables of the group matrix are actually the same, as can be seen by setting $s s_k \equiv s_k$ so that $s_i^{-1} \equiv s_i^{-1} s$.

Varying s, it is clear that the matrices $\Pi_s(I)$ generate a representation of S_n. By a suitable choice of M the matrices $M^{-1}\Pi_s(I)M$ may be taken in completely reduced form, and

$$\begin{aligned} M^{-1}\Pi_s(H)M &= M^{-1}\Pi_I(H)M . M^{-1}\Pi_s(I)M \\ &= M^{-1}\Pi_s(I)M . M^{-1}\Pi_I(H)M. \end{aligned}$$

It follows from Schur's Lemma[12] that, for any element A of S_{mn},

$$3.6 \qquad \operatorname{tr} \Pi_s(A) = \sum_{\beta} \operatorname{tr}(A) \text{ in } [a] \odot [\beta] \,.\, \chi_\beta(s),$$

where $\chi_\beta(s)$ is the character of s in the irreducible representation $[\beta]$ of S_n. The concluding step in this part of the argument is to solve 3.6 by multiplying by $\chi_\beta(s)$ and summing over s to yield

$$3.7 \qquad \chi(A) \text{ in } [a] \odot [\beta] = \frac{1}{n!} \sum_s \operatorname{tr} \Pi_s(A) \,.\, \chi_\beta(s),$$

which can serve to define the reducible representation $[a] \odot [\beta]$ of S_{mn} for the

[11] [1] pp. 22-24 and 231-242.

[12] [10]. This paper was translated and published by Dickson as chapter XI of his *Modern Algebraic Theories* (Chicago, 1926).

time being. Setting $A = I$ in 3.7 we obtain a contribution from the right-hand side only for $s = I$ so, from 3.3, it follows that the

3.8 $$\text{degree of } [\alpha]\odot[\beta] = \frac{(mn)!}{(m!)^n n!}\,(x_\alpha)^n x_\beta,$$

and the frequency of $[\alpha]\odot[\beta]$ in $\Pi_I(H)$ is x_β. Thus

3.9 $$\Pi_n^{[\alpha]} = \sum_\beta x_\beta [\alpha]\odot[\beta].$$

An explicit definition of $[\alpha]\odot[\beta]$ will be given in 6 below.

4. The character of $[\alpha]\odot[\beta]$. It is necessary to return briefly to the abstract theory and consider further the relation of the subgroup H to $\mathfrak{C}(s^*)$ for a given s^* of S^*_n. We can write

4.1 $$H = \mathfrak{C} + \mathfrak{C}g_2 + \ldots + \mathfrak{C}g_r,$$

where $r = (m!)^{n-\omega}$. Now the g_i can be chosen to be the elements of certain of the original S_m's so that

4.2 $$G = \{g_i\} = S_m \times S_m \times \ldots \times S_m, \qquad n-\omega \text{ factors.}$$

In particular, any $\lambda - 1$ of the λ factors S_m linked in a given factor of $\mathfrak{C}$ will appear in G; it is immaterial which factor is omitted. Collecting together these omitted factors, however, we can construct

4.3 $$K = S_m \times S_m \times \ldots \times S_m, \qquad \omega \text{ factors,}$$

and $H = K \times G$.

We are seeking the distribution of the conjugate sets of S_{mn} in Hs, where again we suppose that $s^* = s$. If g is an element of G,

$$g^{-1}Ksg = Kg^{-1}sg.$$

From the definitions of G and K it follows that $sgs^{-1} = gkg'$, where $g' \neq I$ in G and k is an element of K, so that

$$g^{-1}Ksg = Kkg's = Kg's.$$

If we assume that $g_1^{-1}Ksg_1 = g_2^{-1}Ksg_2$ then $s(g_1g_2^{-1})s^{-1} = (g_1g_2^{-1})k$, for which $g' = I$ and this is impossible unless $g_1 = g_2$; this implies that all such conjugate cosets of K are distinct and make up $K \times G = H$. Thus

4.4 *The $(m!)^{n-\omega}$ cosets of K in Hs are conjugate to Ks under transformations by elements of G.*

We are now in a position to proceed to the calculation[10] of tr $\Pi_s(A)$. From 3.5 we have

4.5 $$\text{tr } \Pi_s(H) = \sum r_p^{-1}s_i^{-1}s\mathbf{H}s_ir_p = \sum r_p^{-1}s_i^{-1}\mathbf{H}ss_ir_p,$$

and we obtain a contribution only for elements A of S_{mn} contained in $sH = Hs$. Each of the terms in 4.5 yields the same contribution, and there are $(mn)!/(m!)^n$ terms. Moreover, changing the labels attached to the independent variables of the group matrix in 4.5 can be accomplished by multiplying $\mathbf{H}$ by the operator $\bar{s}$ either on the right or on the left. Thus *the con-*

tribution of each element conjugate to A in Hs is $\chi(A)$, where $\chi(A)$ is the character of A in the representation of $\mathfrak{R}(H)$ generated by $[a]^n$; there are ${n_A}^{H_s}$ such elements. But there are n_A elements conjugate to A in S_{mn}, so to obtain $\operatorname{tr} \Pi_s(A)$ for a given A we must divide by n_A, thus

$$4.6 \qquad \operatorname{tr} \Pi_s(A) = \frac{(mn)!{n_A}^{H_s}}{(m!)^n n_A} \cdot \chi(A).$$

In view of 4.4 we need only consider elements $A = ks$ in calculating $\chi(A)$.

As we saw in 2.6, the structure of $\bar{s}$ depends on $\mathfrak{C}(s)$. The subgroup K was chosen so that every set of symbols which are linked in $\mathfrak{C}(s)$ is represented by just one factor S_m in 4.3. Now the matrix representing a substitution k of K in $[a]^n$ consists of the matrix representing k in $[a]^w$ repeated x_a^{n-w} times down the diagonal. Multiplying by $\bar{s}$ yields a matrix whose diagonal elements are all zeros except for those of k in $[a]^w$, each of which appears once corresponding to the 1's in the diagonal of $\bar{s}$. If $\chi(A)$ is the character of $A = ks$ in $[a]^n$ and $\theta(k)$ is the character of k in $[a]^w$, this proves that

$$4.7 \qquad \chi(A) = \theta(k),$$

and 4.6 becomes

$$4.8 \qquad \operatorname{tr} \Pi_s(A) = \frac{(mn)!{n_A}^{H_s}}{(m!)^n n_A} \cdot \theta(k).$$

In the particular case $[a] = [m]$, the identity representation of S_m, $\theta(k) = 1$ and 4.8 becomes

$$4.9 \qquad \operatorname{tr} \Pi_s(A) = \frac{(mn)!{n_A}^{H_s}}{(m!)^n n_A},$$

which should be compared with Frobenius' formula

$$4.10 \qquad \operatorname{tr} \Pi_I(A) = \frac{(mn)!{n_A}^{H}}{(m!)^n n_A}.$$

Inserting the expression 4.6 in 3.7 we have

$$4.11 \qquad \chi(A) \text{ in } [a]\odot[\beta] = \frac{(mn)!}{(m!)^n n! n_A} \sum_s \chi_\beta(s){n_A}^{H_s}\chi(A);$$

we shall see the significance of this formula in 6 below. Setting $A = I$, it follows that $s = k = I$ and we verify 3.8.

5. The irreducible components of $[a]\odot[\beta]$. In order to obtain the frequency $\mathbf{f}$ with which a given irreducible representation $[\lambda]$ of S_{mn} appears as a component of $[a]\odot[\beta]$ we multiply 4.11 by $\chi_\lambda(A)$ and sum over A to yield

$$5.1 \qquad \mathbf{f}_\lambda([a]\odot[\beta]) = \frac{1}{(m!)^n n!} \sum_s \chi_\beta(s)\,[\chi(hs) \cdot \phi_\lambda(hs)],$$

$$= \frac{1}{(m!)^n n!} \chi_\beta(s) \cdot [\chi(hs) \cdot \phi_\lambda(hs)].$$

Here we are denoting by $\phi_\lambda(hs)$ a vector whose $(m!)^n$ components are the

characteristics of the elements hs in the representation $[\lambda]$ of S_{mn}; similarly, $\chi(hs)$ denotes a vector whose components are the characteristics of the elements hs in the representation of $\mathfrak{N}(H)$ generated by $[\alpha]^n$ and calculated as described in the preceding section. The dot denotes scalar multiplication.

In the case $[\beta] = [n]$, which is important for the theory of invariants, $\chi_\beta(s) = 1$ and 5.1 simplifies somewhat to give

$$5.2 \qquad f_\lambda([\alpha]\odot[n]) = \frac{1}{(m!)^n n!} \sum_s \chi(hs) \cdot \phi_\lambda(hs).$$

If we also set $[\alpha] = [m]$, 5.2 reduces to the familiar formula[13] giving the frequency with which $[\lambda]$ appears as an irreducible component of the representation of S_{mn} induced by the identity representation of $\mathfrak{N}(H)$; this representation is in fact the permutation representation to which the subgroup $\mathfrak{N}(H)$ gives rise under right multiplication by the elements of S_{mn}.

It is interesting to note the effect of changing the representations $[\alpha]$ and $[\beta]$ into their conjugates,[14] that is, interchanging the rows and columns of the corresponding Young diagrams. It is clear that we must distinguish two cases according as m is even or odd. If m is even every element of S^*_n, considered as a permutation on mn symbols, is even. On the other hand, if m is odd $\frac{1}{2}(n!)$ of the permutations of S^*_n are even and $\frac{1}{2}(n!)$ are odd. Changing $[\alpha]$ into $[\alpha']$ will change the sign of the component of $\chi = \theta$ associated with an odd permutation $h = k$, but this will be compensated for in 5.1 by changing $[\lambda]$ into $[\lambda']$, provided m is *even*; if m is *odd* ks^* may be odd or even depending on s^*, and to achieve compensation it will be necessary also to change $[\beta]$ into $[\beta']$. This yields D. E. Littlewood's[9]

5.3 *Theorem of Conjugates*:

$$f_\lambda([\alpha]\odot[\beta]) = f_{\lambda'}([\alpha']\odot[\beta]), \ m \text{ even},$$
$$= f_{\lambda'}([\alpha']\odot[\beta']), \ m \text{ odd}.$$

6. Explicit definition of $[\alpha]\odot[\beta]$. Let us return to the representation of $\mathfrak{N}(H)$ generated by $[\alpha]^n$ as defined in **2**. Clearly, the representation $[\beta]$ of S_n is also an irreducible representation of $\mathfrak{N}(H)$ in which every element of H is represented by the identity matrix. These are the two extreme members of a family of representations of $\mathfrak{N}(H)$ which we can easily construct. The remaining irreducible representations of $\mathfrak{N}(H)$ are similar in character, but we do not need to refer to them here.

If we denote by $s(\beta)$ the matrix representing s^* in the representation $[\beta]$, then the matrices obtained by replacing each 1 of $\bar{s}$ by $s(\beta)$ can be written

$$6.1 \qquad \bar{s} \times s(\beta),$$

and these also yield a representation of S^*_n of degree $x_\alpha^n \cdot x_\beta$. Constructing the analogous representation

$$6.2 \qquad [\alpha]^n \times I_{x_\beta}$$

[13] [13] p. 287.
[14] [12] p. 202.

of H, and combining 6.1 and 6.2 we obtain a representation of $\mathfrak{R}(H)$ which we may denote by $[\alpha, \beta]$; it follows from Schur's Lemma that $[\alpha, \beta]$ is irreducible. The representation of $\mathfrak{R}(H)$ defined in **2**, for which $[\beta] = [n]$ is here denoted $[\alpha, n]$, and the representation $[\beta]$ of $\mathfrak{R}(H)$ is denoted $[m, \beta]$. The character of any substitution hs^* of $\mathfrak{R}(H)$ in $[\alpha, \beta]$ is clearly $\chi(hs^*) \cdot \chi_\beta(s^*)$; thus, substituting in Frobenius' formula[16] for the character of an induced representation, we obtain 4.11 and conclude that

6.3 $[\alpha]\odot[\beta]$ *is that representation of* S_{mn} *which is induced by the irreducible representation* $[\alpha, \beta]$ *of* $\mathfrak{R}(H)$.

7. Illustrative examples. Referring to the example of 1, we calculate the irreducible components[15] of $[2, 1]\odot[2]$. Here K and G are interchangeable; set

$$7.1 \qquad K = I + (12) + (13) + (23) + (123) + (132),$$

so that

$$7.2 \qquad Hs = Ks + \text{five similar cosets of } K,$$

and

$$7.3 \qquad Ks = (14)(25)(36) + (1425)(36) + (1436)(25) + (2536)(14) + (142536) + (143625).$$

With the help of a table of characters S_6 we obtain from 5.1 :

$$7.4 \qquad \begin{aligned} [2, 1]\odot[2] &= [4, 2] + [3, 2, 1] + [3, 1^3] + [2^3], \\ [2, 1]\odot[1^3] &= [2^2, 1^2] + [3, 2, 1] + [4, 1^2] + [3^2], \end{aligned}$$

both of degree 40, illustrating 5.3 with m odd.

In order to illustrate the case with m even consider the reduction of the disjoint product $[2]\cdot[2]\cdot[2]$. The subgroups H and S^*_3 are easily seen to be:

$$7.5 \qquad H = \{I + (12)\} \times \{I + (34)\} \times \{I + (56)\},$$

$$7.6 \qquad S^*_3 = I + (13)(24) + (15)(26) + (35)(46) + (135)(246) + (153)(264).$$

It is only necessary to write down two cosets Hs^* since S^*_3 has only two conjugate sets distinct from the identity:

$$7.7 \qquad H(13)(24) = (13)(24) + (1324) + (1423) + (13)(24)(56) + (14)(23) + (1324)(56) + (1423)(56) + (14)(23)(56),$$

$$7.8 \qquad H(135)(246) = (135)(246) + (135246) + (146235) + (136245) + (146)(235) + (136)(245) + (145)(236) + (145236).$$

From 5.1 we have:[16]

$$7.9 \qquad \begin{aligned} [2]\odot[3] &= [6] + [4, 2] + [2^3], && \text{of degree 15}, \\ [2]\odot[2, 1] &= [5, 1] + [4, 2] + [3, 2, 1], && \text{of degree 30}, \\ [2]\odot[1^3] &= [4, 1^2] + [3^2], && \text{of degree 15}. \end{aligned}$$

[15] [3] p. 336.
[16] [4] p. 289.

An application of 5.3 yields:

$$
\begin{aligned}
&[1^3]\odot[3] &&= [1^6] + [2^2, 1^2] + [3^2], && \text{of degree 15,}\\
7.10\qquad &[1^3]\odot[2,1] &&= [2,1^4] + [2^2,1^2] + [3,2,1], && \text{of degree 30,}\\
&[1^3]\odot[1^3] &&= [3,1^3] \;+\; [2^3], && \text{of degree 15.}
\end{aligned}
$$

Once the necessary cosets have been written out the irreducible components of any $[\alpha]\odot[\beta]$ can be obtained by purely arithmetic calculations, assuming that the necessary tables of characters are available.

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The University of Toronto

BOUNDEDNESS PROPERTIES IN FUNCTION-LATTICES

M. H. STONE

THE continuous real functions on a topological space X are partially ordered in a natural way by putting $f \leqq g$ if and only if $f(x) \leqq g(x)$ for all x in X. With respect to this partial ordering these functions constitute a lattice, the lattice operations $\cup$ and $\cap$ being defined by the relations $(f \cup g)(x) = \max(f(x), g(x))$, $(f \cap g)(x) = \min(f(x), g(x))$. The lattice character of any partially ordered system merely expresses the existence of least upper and greatest lower bounds for any finite set of elements in the system. Many partially ordered systems enjoy much stronger boundedness properties than these: for example, every subset with an upper bound may have a least upper bound, as in the case of the real number system. It is our purpose in this paper to determine the conditions under which the function-lattice described above exhibits behaviour of this kind. As an illustration of the results established we may take the important case where X is a compact Hausdorff space: here we find that the boundedness property required of the function-lattice is equivalent to the condition that X be the Boolean space associated with a Boolean algebra which has a certain strong additivity property; and many curious and useful relationships among the functions of Baire on the space X appear as a consequence of this condition, which implies that each bounded function of Baire on X differs from a uniquely determined continuous function only on a set of the first category.[1]

1. The spectral analysis of a real function. We shall first lay the foundations for the main investigation by developing what may be called the spectral analysis of a real function. We denote by capital Latin letters the abstract set X and its subsets, by lower case Latin letters real-valued functions on X, by capital Greek letters the real number system Σ and its subsets, and by lower-case Greek letters real numbers. At certain points we shall suppose that X is a topological space and the functions on it continuous. We first make the following observation.

Received May 6, 1948.

[1]The results of this paper, for the case where X is compact, were announced in general terms and without proof in the Proceedings of the National Academy of Sciences, vol. 26 (1940), 280-283; and important applications were indicated there. Full discussions for the case where X is a completely regular space satisfying the category condition (C) of Sec. 2 were presented in lectures at Harvard, Brown, Chicago, and the Universidad del Litoral (Rosario, Argentina) in 1942 and 1943. As referee, Professor I. Halperin pointed out that it would be desirable to isolate and minimize the rôle played by the condition (C). His detailed suggestions to this end have been incorporated in Sec. 2.

THEOREM 1. *The set* $E_f(\lambda) = \{x; f(x) < \lambda\} \subset X$ *increases with* λ *and enjoys the properties*

$$\bigcap_{\lambda \in \Sigma} E_f(\lambda) = O, \quad \bigcup_{\lambda \in \Sigma} E_f(\lambda) = X, \quad \bigcup_{\mu < \lambda} E_f(\mu) = E_f(\lambda),$$

$$\{x; f(x) = \lambda\} = P_f(\lambda) \text{ where } P_f(\lambda) = E_f(\lambda)' \cap \bigcap_{\mu > \lambda} E_f(\mu),$$

(and even the equivalent properties in which the operations $\bigcap$, $\bigcup$ *affect only variables* λ, μ *which belong to a set* Δ *everywhere dense in* Σ*). The characteristic function of* $E_f(\lambda)$ *is expressed by the formula* $\phi_\lambda \times f$, *where* $\phi_\lambda(\mu)$ *is* 1 *or* 0 *according as* $\mu < \lambda$ *or* $\mu \geqq \lambda$ *and* $(\phi_\lambda \times f)(x) = \phi_\lambda(f(x))$.

The family of sets $E_f(\lambda)$ may be called the *spectral family* for the function f. The formal analogy between this family and the spectral family for a self-adjoint operator in Hilbert space explains our choice of terminology. Eventually the analogy proves to be much more than formal, though we shall not discuss this circumstance in the present paper.

It is now a simple task to invert the theorem above.

THEOREM 2. *If* $E(\lambda)$, $-\infty < \lambda < +\infty$, *has the properties*

$$\bigcap_{\lambda \in \Sigma} E(\lambda) = O, \quad \bigcup_{\lambda \in \Sigma} E(\lambda) = X, \quad \bigcup_{\mu < \lambda} E(\mu) = E(\lambda)$$

(or even the equivalent properties in which the operations $\bigcap$, $\bigcup$ *affect only variables* λ, μ *which belong to a set* Δ *everywhere dense in* Σ*), then* $E(\lambda)$ *is the spectral family of a function* f *defined by putting* $f(x) = \lambda$ *when and only when* x *is in* $P(\lambda) = E(\lambda)' \cap \bigcap_{\mu > \lambda} E(\mu)$.

Proof. First let us demonstrate that the non-void sets $P(\lambda)$ constitute a partition of X—in other words, that the relations

$$P(\lambda_1) \cap P(\lambda_2) = O \text{ when } \lambda_1 \neq \lambda_2, \quad \bigcup_{\lambda \in \Sigma} P(\lambda) = X$$

hold. In the typical case where $\lambda_1 < \lambda_2$, the first relation follows immediately from the relations $P(\lambda_1) \subset E(\lambda_2)$, $P(\lambda_2) \subset E(\lambda_2)'$. The other is not quite so easy to prove. If x is a fixed element in X, the relations $x \in E(\mu)$ and x non $\in E(\mu)$ define respectively the upper and lower sections of a Dedekind cut in Σ by virtue of the fact that $\mu_1 < \mu_2$ implies $E(\mu_1) \subset E(\mu_2)$. Let $\lambda(x)$ be the real number determining this cut. The relation $\bigcup_{\mu < \lambda(x)} E(\mu) = E(\lambda(x))$ shows that x non $\in E(\lambda(x))$ and hence that $\lambda(x)$ belongs to the lower section. We conclude that $x \in E(\lambda(x))' \cap \bigcap_{\mu > \lambda(x)} E(\mu) = P(\lambda(x))$, $\bigcup_{x \in X} P(\lambda(x)) = X$. We also see that a real function f can be defined in the manner described in the theorem, and that this function is indeed given also by the relation $f(x) = \lambda(x)$. In order to verify that $E(\lambda) = E_f(\lambda)$, we note that $f(x) = \mu < \lambda$ implies $x \in P(\mu) \subset E(\lambda)$ while $x \in E(\lambda)$ implies $x \in E(\mu)$ for $\mu \geqq \lambda$ and hence x non$\in P(\mu) \subset E(\mu)'$ for $\mu \geqq \lambda$, a relation which requires that $f(x) < \lambda$.

A result useful in applications may be noted here.

THEOREM 3. *If $H(\lambda) \subset X$ is defined for all λ in an everywhere dense subset Λ of Σ and has the properties $\bigcap_{\lambda \in \Lambda} H(\lambda) = O$, $\bigcup_{\lambda \in \Lambda} H(\lambda) = X$, $H(\lambda_1) \subset H(\lambda_2)$ when $\lambda_1 < \lambda_2$, then the set $E(\lambda) = \bigcup_{\mu \in \Lambda,\ \mu < \lambda} H(\mu)$, $-\infty < \lambda < +\infty$, has the properties enumerated in Theorem 2. In order that $E(\lambda) = H(\lambda)$ for $\lambda \in \Lambda$ it is necessary and sufficient that $H(\lambda) = \bigcup_{\mu \in \Lambda,\ \mu < \lambda} H(\mu)$ for $\lambda \in \Lambda$.*

Various simple properties of functions are reflected by simple properties of their spectral families. Thus we see immediately that the following statements are true.

THEOREM 4. *In order that $\alpha \leqq f(x) \leqq \beta$ for all x in X it is necessary and sufficient that $E_f(\lambda) = O$ for $\lambda \leqq \alpha$ and $E_f(\lambda) = X$ for $\lambda > \beta$.*

THEOREM 5. *In order that $f \leqq g$ it is necessary and sufficient that $E_f(\lambda) \supset E_g(\lambda)$ for all λ.*

When X is a topological space there are connections between topological properties of a function and topological properties of its spectral family. A very useful instance of this general remark is the following proposition.

THEOREM 6. *When X is a topological space, a necessary and sufficient condition for the continuity of the function f is that $E_f(\lambda_1)$ be strongly contained in $E_f(\lambda_2)$ for $\lambda_1 < \lambda_2$—in other words, that the closure of $E_f(\lambda_1)$ be interior to $E_f(\lambda_2)$: $E_f(\lambda_1)^- \subset E_f(\lambda_2)'^{-\prime}$.*

Proof. When f is continuous, the sets $E_f(\lambda_1)$ and $E_f(\lambda_2)$ are open while the set $\{x; f(x) \leqq \frac{1}{2}(\lambda_1 + \lambda_2)\}$ is closed. When $\lambda_1 < \lambda_2$ the relations $E_f(\lambda_1) \subset \{x; f(x) \leqq \frac{1}{2}(\lambda_1 + \lambda_2)\} \subset E_f(\lambda_2)$ hold, and the condition of the theorem follows from them. On the other hand, when this condition holds, the relations $E_f(\lambda) = \bigcup_{\mu < \lambda} E_f(\mu) \subset \bigcup_{\mu < \lambda} E_f(\mu)^- \subset \bigcup_{\mu < \lambda} E_f(\frac{1}{2}(\lambda + \mu))'^{-\prime} \subset \bigcup_{\mu < \lambda} E_f(\frac{1}{2}(\lambda + \mu)) \subset E_f(\lambda)$ are valid and show that $E_f(\lambda) = \bigcup_{\mu < \lambda} E_f(\frac{1}{2}(y + \mu))'^{-\prime}$ is an open set. In the same way the set $H_f(\lambda) = \{x; f(x) > \lambda\}$ is seen to be open, since $H_f(\lambda) = \bigcup_{\mu > \lambda} E_f(\mu)' \subset \bigcup_{\mu > \lambda} E_f(\mu)'^{-} \subset \bigcup_{\mu > \lambda} E_f(\frac{1}{2}(\lambda + \mu))^{-\prime} \subset \bigcup_{\mu > \lambda} E_f(\frac{1}{2}(\lambda + \mu))' \subset H_f(\gamma)$. Thus the set $\{x; \alpha < f(x) < \beta\} = H_f(\alpha) \cap E_f(\beta)$ is open and the function f is continuous.

In actual constructions the following variant of Theorem 3 is often useful.

THEOREM 7. *If, when X is a topological space, the set $H(\lambda) \subset X$ is defined for all λ in an everywhere dense subset Λ of Σ and has the properties stated in Theorem 3 with the modification that $H(\lambda_1)$ is strongly contained in $H(\lambda_2)$ for $\lambda_1 < \lambda_2$, then the family of sets $E(\lambda) = \bigcup_{\mu \in \Lambda,\ \mu < \lambda} H(\mu)$ is the spectral family for a continuous function f.*

Proof. We have to verify that $E(\lambda) = E(\lambda)$ satisfies the condition of the preceding theorem. When λ_1 and λ_2 are given, $\lambda_1 < \lambda_2$, we choose μ_1 and μ_2 in Λ so that $\lambda_1 < \mu_1 < \mu_2 < \lambda_2$. We then have $E(\lambda_1) \subset H(\mu_1)$, $H(\mu_2) \subset E(\lambda_2)$ and hence $E(\lambda_1)^- \subset H(\mu_2)^- \subset H(\mu_2)'^{-\prime} \subset E(\lambda_2)'^{-\prime}$, as we wished to show.

We may digress briefly to show how Theorem 7 yields the crucial property of normal spaces—namely, the existence of a continuous function f vanishing throughout one closed set of a disjoint pair A, B and taking on the value 1 throughout the other. We choose Λ as the set of all dyadically rational numbers; and for $\lambda \epsilon \Lambda$, we put $H(\lambda) = O$, $H(\lambda) = A$, $H(\lambda) = B'$, $H(\lambda) = X$ according as $\lambda < 0, \lambda = 0, \lambda = 1, \lambda > 1$. We then determine $H(\lambda)$ for $\lambda = \frac{p}{2^n}$, $0 \leqq \lambda \leqq 1$, by induction on n, starting from the case $n = 0$: if we have found $H\left(\frac{p}{2^n}\right)$ for $p = 0, 1, 2, \ldots, 2^n$ so that $H\left(\frac{p}{2^n}\right)$ is strongly contained in $H\left(\frac{p+1}{2^n}\right)$, we can then choose $H\left(\frac{2p+1}{2^{n+1}}\right)$ as an open set containing $H\left(\frac{p}{2^n}\right)^-$ and strongly contained in $H\left(\frac{p+1}{2^n}\right)^{\prime -}$, by virtue of the normality of the space X. Theorem 7 can now be applied to the family $H(\lambda)$, and shows that the required function f exists. Moreover $0 \leqq f(x) \leqq 1$ for all x in X in accordance with Theorem 4.

We close this section with a result of some intrinsic interest in the light of later investigations.

THEOREM 8. *In order that, when X is a topological space, the set $\{x; f(x) \neq g(x)\}$ be of the first category (that is, be the union of a denumerable family of nowhere dense sets) it is necessary and sufficient that the symmetric difference $E_f(\lambda) + E_g(\lambda) = (E_f(\lambda)' \cap E_g(\lambda)) \cup (E_f(\lambda) \cap E_g(\lambda)')$ be a set of first category for every λ. Indeed this condition remains sufficient if λ be restricted to an everywhere dense subset Λ of Σ.*

Proof. It is clear that $A \subset \{x; f(x) = g(x)\}$ if and only if $E_f(\lambda) \cap A = E_g(\lambda) \cap A$ for all λ. Thus if $A \subset \{x; f(x) = g(x)\}$ and $B = A' \supset \{x; f(x) \neq g(x)\}$, where B is of the first category, we must have $B \supset E_f(\lambda) + E_g(\lambda)$ and can conclude that the latter set is of the first category. On the other hand, if $E_f(\lambda) + E_g(\lambda)$ is of the first category for all λ in an everywhere dense subset Λ of Σ, we determine a denumerable everywhere dense subset $\mathbf{M}$ of Λ and put $B = \bigcup_{\lambda \epsilon \mathbf{M}} (E_f(\lambda) + E_g(\lambda))$. Here B is clearly a set of the first category. If $A = B'$ we have $E_f(\lambda) \cap A = E_g(\lambda) \cap A$ for $\lambda \epsilon \mathbf{M}$. However this means that we must also have for arbitrary λ

$$E_f(\lambda) \cap A = \bigcup_{\mu \epsilon \mathbf{M},\, \mu < \lambda} E_f(\mu) \cap A = \bigcup_{\mu \epsilon \mathbf{M},\, \mu < \lambda} E_g(\mu) \cap A = E_g(\lambda) \cap A$$

by virtue of Theorem 1. Accordingly we see that $B \supset \{x; f(x) \neq g(x)\}$ and the latter set is of the first category.

2. The main investigation. Let $C(X)$ be the partially ordered system of all continuous real functions on the topological space X. Of course, $C(X)$ is a lattice. The following stronger boundedness conditions are, as is well known, equivalent:

($A_{\aleph}$) *every non-void part of $C(X)$ which has an upper bound and at most $\aleph$ members has a least upper bound;*

($B_{\aleph}$) *every non-void part of $C(X)$ which has a lower bound and at most $\aleph$ members has a greatest lower bound.*

Naturally if the cardinal number of $C(X)$ does not exceed $\aleph$ these conditions become respectively:

(A) *every non-void part of $C(X)$ which has an upper bound has a least upper bound;*

(B) *every non-void part of $C(X)$ which has a lower bound has a greatest lower bound.*

It is our object to determine topological properties of the space X which are equivalent to ($A_{\aleph}$) and ($B_{\aleph}$). In doing so there is no essential loss of generality in assuming that X is completely regular: for the study of the real continuous functions on a general topological space X can always be reduced to the study of corresponding continuous functions on a certain completely regular continuous image of X which is a topological invariant of X. On the other hand, the interests of simplicity are often served by making this assumption, as we shall do here at our convenience.

At certain points it will also be convenient to introduce the hypothesis that

(C) *every non-void open subset of X is of the second category (i.e. is not of the first category).*

A simple condition for a completely regular space X to have the property (C) is that X be locally $\aleph_0$-compact in the following sense: every point of X has such a neighbourhood U that the relations $\bigcap_{n=1}^{\infty} X_n = O$, $X_n \subset U$ for closed sets X_n imply $X_1 \cap \ldots \cap X_m = O$ for some m. We sketch the proof. Let Y be a non-void open set, $\{Y_n\}$ a sequence of nowhere dense sets. For a point of Y let U be such a neighbourhood as was described above. Let $X_0 = X$, and suppose that closed sets $X_1, \ldots, X_{n-1}$ have been determined so that $O \neq X_k'^{-\prime} \subset X_k \subset Y \cap U \cap Y_1^{-\prime} \cap \ldots \cap Y_k^{-\prime} \cap X_{k-1}'^{-\prime}$ for $k = 1, \ldots n-1$. Since Y_n is nowhere dense the open set $Y_n^{-\prime} \cap X_{n-1}'^{-\prime}$ is non-void. Hence, by the complete regularity of X, there exists a continuous function f_n, $0 \leqq f_n \leqq 1$, which vanishes at some point of this open set and assumes the value 1 on its complement. The set $X_n = \{x; f_n(x) \leqq \frac{1}{2}\}$ is then a closed set such that $O \neq X_n'^{-\prime} \subset X_n \subset Y \cap U \cap Y_1^{-\prime} \cap \ldots \cap Y_n^{-\prime} \cap X_{n-1}'^{-\prime}$. Since $X_1 \cap \ldots \cap X_m = X_m \neq O$ and $X_n \subset U$, we see that $\bigcap_{n=1}^{\infty} X_n \neq O$ and hence that Y contains at least one point not in $\bigcup_{n=1}^{\infty} Y_n$.

We now assume that the equivalent conditions ($A_{\aleph}$), ($B_{\aleph}$) are satisfied for a fixed infinite cardinal number $\aleph$ and a given completely regular space X; and we investigate the consequences for X and $C(X)$. As a first step let $\{f_n\}$ be a sequence in $C(X)$ with a lower bound—and hence, by hypothesis, a greatest

lower bound, f. The sequence of functions $g_n = \min(f_1 - f, \ldots, f_n - f)$ is in $C(X)$, it decreases with $1/n$, and it has in $C(X)$ the greatest lower bound 0 (the constant everywhere equal to zero). Since $g_n(x)$ decreases with $1/n$, and is non-negative, the limit $g(x) = \lim_{n\to\infty} g_n(x)$ exists and defines a function $g \geqq 0$ which is not necessarily continuous. We now observe that the set $\{x; g(x) > 0\}$ is expressible in the form $\bigcup_{p=1}^{\infty} X_p$ where $X_p = \{x; g(x) \geqq 1/p\} = \bigcap_{n=1}^{\infty} \{x; g_n(x) \geqq 1/p\}$, and is therefore an F_σ-set—that is, is the union of a sequence of closed sets. If the interior of X_p were non-void, we could derive a contradiction, as follows. Since X is completely regular, the existence of a point x_0 in ${X_p}'^{-\prime}$ would imply the existence of a function h belonging to $C(X)$, bounded between 0 and 1, assuming the value 1 at x_0, and vanishing on the closed set ${X_p}'^{-}$. It would then be true that $g \geqq (1/p)h$ and hence that $g_n \geqq (1/p)h$ for $n = 1, 2, 3, \ldots$. Consequently, $0 \geqq (1/p)h$—in contradiction with the fact that $(1/p)h(x_0) = 1/p > 0$. Inasmuch as X_p is closed and contains no interior point it must be nowhere dense. Hence the set $\{x; g(x) > 0\}$ is a set of the first category. In terms of the original functions f_n, f, this means that the continuous function f is uniquely determined as the maximum element in $C(X)$ which is everywhere less than or equal to $\inf_n f_n(x)$; and that the set $\{x; f(x) < \inf_n f_n(x)\}$ is of the first category. From this general result we can now infer that, if the sequence of continuous functions f_n is bounded in $C(X)$, then $\limsup_{n\to\infty} f_n$ (and similarly $\liminf_{n\to\infty} f_n$) exists and differs from a continuous function only on a set of first category. Indeed, the result just established shows that $\lim_{n\to\infty} \max(f_n, \ldots, f_{n+p}) \leqq g_n$ where g_n is the minimal element in $C(X)$ with this property; and that $Y_n = \{x; \lim_{p\to\infty} \max(f_n, \ldots, f_{n+p}) < g_n\}$ is a set of the first category. Moreover the characterization of g_n shows that $\lim_{p\to\infty} \max(f_m, \ldots, f_{m+p}) \leqq g_n$ when $m \geqq n$ and hence that $g_m \leqq g_n$. The lower bounds of $\{f_n\}$ are also lower bounds of $\{g_n\}$. Now $\lim_{n\to\infty} g_n$ exists and differs from a continuous function f only on a set of the first category; and at the same time $\lim_{n\to\infty} g_n$ and $\limsup_{n\to\infty} f_n$ differ only on the set $Y = \bigcup_{n=1}^{\infty} Y_n$, which is of the first category. Hence $\limsup_{n\to\infty} f_n$ differs from f, a continuous function, only on a set of the first category. As a corollary of the facts established above, we remark that if a sequence $\{f_n\}$ bounded in $C(X)$ converges everywhere then its limit differs only on a set of the first category from a continuous function which is bounded by the bounds of $\{f_n\}$. In fact we can generalize this result to the family of all functions of Baire on X, as follows:

THEOREM 9. *For a completely regular space X the conditions $(A_\aleph)$, $(B_\aleph)$ imply that every real function of Baire which is bounded by continuous functions*

differs only on a set of the first category from a continuous function (which under the hypothesis (C) is uniquely determined); and these conditions further imply that every (finite) real function of Baire differs only on a set of the first category from the quotient of two continuous functions (with denominator vanishing only on a nowhere dense set when hypothesis (C) is valid).

Proof. It is necessary to proceed by transfinite induction based on the classical procedure of Baire. Let Ω be the first ordinal number such that the class of all preceding ordinal numbers is non-denumerable. The class of all ordinal numbers α such that $0 \leqq \alpha < \Omega$ is the substratum for the recursive definition of the Baire classes: the continuous functions on X are taken as constituting the 0th Baire class; and, when the βth Baire class has been defined for every $\beta < \alpha < \Omega$, then those functions which are limits of sequences of functions in the available classes without being members of any such class are taken as constituting the αth Baire class. The union of all the Baire classes is the family of Baire functions. We have already seen above that every function of the first Baire class which is bounded with respect to $C(X)$ differs from a continuous function only on a set of the first category. Suppose we have proved the like result for every function in the βth Baire class, for $\beta < \alpha$. Let f be in the αth Baire class—specifically let f_n be in the β_nth Baire class, $\beta_n < \alpha$, and let the sequence $\{f_n\}$ be bounded with respect to $C(X)$ and converge to f. By hypothesis there exists a continuous function g_n which differs from f_n only on a set of the first category. We may suppose that the sequence $\{g_n\}$ is bounded in $C(X)$: otherwise we could replace g_n by the function max $(h, \min(g_n, k))$ where h and k are continuous functions such that $h \leqq f_n \leqq k$ for all n. Thus we see that $\limsup_{n\to\infty} g_n$ exists and differs from $f = \lim_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$ only on a set of the first category. The results proved above show that there is a continuous function g which differs from $\limsup_{n\to\infty} g_n$, and hence also from f, only on a set of the first category. Now (C) implies the uniqueness of g: for if g_1 and g_2 are continuous functions each differing from f only on a set of the first category, then the set $\{x; g(x_1) \neq g(x_2)\}$ is open and of the first category, hence void; and $g_1 = g_2$. The part of the theorem dealing with functions of Baire which are bounded by continuous functions now follows in accordance with the principle of transfinite induction. When f is a general (finite) function of Baire, the functions $f/(1+f^2)$ and $1/(1+f^2)$ are bounded functions of Baire, equal except on a set of the first category to continuous functions g and h respectively. It is evident that $f = g/h$ except on a set of the first category. When (C) holds we see that the closed set $\{x; h(x) = 0\}$, being of the first category, has void interior and is therefore nowhere dense.

An immediate consequence of the property just established for the Baire functions is this:

THEOREM 10. *Let ϕ be any real function of Baire of the real variable λ on a domain which includes the range of a given continuous function f on a completely*

regular space X; and let the function $\phi \times f$, where $(\phi \times f)(x) = \phi(f(x))$, be bounded in $C(X)$. Then the conditions $(A_{\aleph})$, $(B_{\aleph})$ imply that $\phi \times f$ differs only on a set of the first category from a continuous function (which is uniquely determined under hypothesis (C)).[3]

The next step in our investigation depends essentially upon a particular case of Theorems 9 and 10.

THEOREM 11. *For a completely regular space X, the conditions $(A_{\aleph})$, $(B_{\aleph})$ imply that the spectral family $\{E_f(\lambda)\}$ for a continuous real function f on X has the properties:*

(1) *$E_f(\lambda)^-$ is open as well as closed;*

(2) *$E_f(\lambda)$ is expressible as the union of a sequence of closed-and-open sets, through the specific formula $E_f(\lambda) = \bigcup_{\mu \in \Lambda,\ \mu < \lambda} E_f(\lambda)^-$, where Λ is a denumerable everywhere dense subset of Σ.*

Proof. The function ϕ_λ defined in Theorem 1 is a function of Baire, being the limit of the increasing sequence of continuous functions $\phi_{\lambda,n}$ where $\phi_{\lambda,n}(\mu) = 1$ for $\mu \leqq \lambda - 1/n$, $\phi_{\lambda,n}(\mu) = n(\lambda - \mu)$ for $\lambda - 1/n \leqq \mu \leqq \lambda$, and $\phi_{\lambda,n}(\mu) = 0$ for $\mu \geqq \lambda$. The function $\phi_\lambda \times f$ is therefore equal except on a set of the first category to a continuous function g which can be taken as the least upper bound in $C(X)$ of the family of continuous functions $\phi_{\lambda,n} \times f$. For this function it is evident that $0 \leqq g \leqq 1$ and that $g(x) = 1$ when $x \in E_f(\lambda)$. The continuity of g implies that $g(x) = 1$ for $x \in E_f(\lambda)^-$. Now let $x_0 \in E_f(\lambda)^{-\prime}$. In accordance with the fact that X is completely regular, there exists a continuous function h such that $0 \leqq h \leqq 1$, $h(x_0) = 0$, $h(x) = 1$ for $x \in E_f(\lambda)^-$. The inequality $\phi_{\lambda,n} \times f \leqq h$ is valid and implies that $g \leqq h$. In particular, $g(x_0) = 0$. Thus the continuous function g is a characteristic function, being equal to 1 on $E_f(\lambda)^-$ and to 0 on its complement; and both these sets are therefore closed-and-open sets. The remainder of the present theorem follows from the relations $E_f(\lambda) = \bigcup_{\mu \in \Lambda,\ \mu < \lambda} E_f(\mu)$, $E_f(\mu)^- \subset E_f(\lambda)$, noted in Theorems 1 and 6.

Concerning the subsets of X which are both closed and open we can now make the following general statements.

THEOREM 12. *For a completely regular space the conditions $(A_{\aleph})$, $(B_{\aleph})$ imply that the closed-and-open subsets of X are an additive basis for the open subsets of X, and that they are an $\aleph$-additive family in the following sense: the union of at most $\aleph$ closed-and-open sets has a closure which is both closed and open.*

Proof. If U is an open set and x_0 a point of U, there exists a continuous function f bounded between 0 and 1 which vanishes at x and assumes the value 1 on U'. The set $E_f(\lambda)^-$ is closed-and-open, in accordance with Theorem 11. Moreover $x \in E_f(\frac{1}{2})^- \subset \{x; f(x) \leqq \frac{1}{2}\} \subset U$. It follows that U is the union of closed-and-open sets. In order to establish the $\aleph$-additivity asserted above,

[3] The proof is supplied by observing that $\phi \times f$ is a function of Baire. For implications of this theorem, see Stone, footnote 4.

we consider the union G of a family of closed-and-open sets G_ρ, where the index ρ belongs to a class of cardinal number at most $\aleph$. The characteristic function g for G is defined in terms of the characteristic functions g_ρ for the sets G_ρ by the formula $g(x) = \sup_\rho g_\rho(x)$. The functions g_ρ are continuous and bounded by the constant 1. Hence they have in $C(X)$ a least upper bound, f. Obviously the relations $g_\rho \leqq f$ imply $g(x) \leqq f(x)$ for every x; in particular $0 \leqq f(x)$ everywhere and $1 \leqq f(x)$ for $x \in G$. Since $g_\rho \leqq 1$, $g_\rho \leqq f$, we must have $g_\rho \leqq \min(1, f)$ and hence $f \leqq \min(1, f) \leqq 1$. It follows that $f(x)$ is equal to 1 for every x in G—and by continuity even for every x in G^-. On the other hand, we can show that $f(x)$ vanishes for every x in the open set $U = G^{-\prime}$. Let x_0 be in U. Then there exists a continuous function h bounded between 0 and 1, vanishing at x_0, and assuming the value 1 on $U' = G^-$. Since $g_\rho \leqq g \leqq h$ for every ρ, we see that $g_\rho \leqq \min(f, h) \leqq f$ for every ρ and hence that $f \leqq \min(f, h)$. Thus $f = \min(f, h)$ and $0 \leqq f(x_0) = \min(f(x_0), h(x_0)) \leqq h(x_0) = 0$, $f(x_0) = 0$. We now see that f is the characteristic function of G^-, this set therefore being open as well as closed. This completes the proof of the theorem.

By combining some of the necessary conditions obtained in Theorems 9-12 we can now exhibit a set of conditions on X and $C(X)$ which implies $(A_\aleph)$ and $(B_\aleph)$ and which is therefore equivalent to each of the latter conditions taken separately. Indeed, some of the properties assumed for the space X are consequences of this set of conditions, so that we can even state a somewhat stronger theorem, as follows.

THEOREM 13. *If X is a topological space, the conditions*

(1) *the closed-and-open sets in X have the $\aleph$-additivity property of the preceding theorems;*

(2) *for any continuous function f, the spectral set $E_f(\lambda)$ is the union of at most $\aleph$ closed-and-open sets*

jointly imply that X has the properties $(A_\aleph)$ and $(B_\aleph)$; and also that (2) actually assumes the stronger form stating that $E_f(\lambda)$ is the union of at most $\aleph_0$ closed-and-open sets. If X is completely regular, then (2) implies that the closed-and-open sets constitute an additive basis for X (a condition which is sufficient to imply complete regularity).

Proof. The final assertion of the theorem requires no detailed examination. Clearly (1) and (2) together imply that $E_f(\lambda)^-$ is open as well as closed. As in the proof of Theorem 11, it then follows that (2) assumes the stronger form stating that $E_f(\lambda)$ is the union of at most $\aleph_0$ closed-and-open sets. Now let us consider a family of at most $\aleph$ continuous functions f_ρ with a lower bound in $C(X)$. The function h such that $h(x) = \inf_\rho f_\rho(x)$ is then defined for all x in X but is not necessarily a continuous function. It is clear that $E_h(\lambda) = \bigcup_\rho E_{f_\rho}(\lambda)$. Hence $E_h(\lambda)$ is the union of at most $\aleph$ closed-and-open sets by (2); and its

closure is an open set by (1). Putting $E(\lambda) = \bigcup_{\mu<\lambda} E_h(\mu)^-$, we see that $E(\lambda) \subset E_h(\lambda)^-$. If $\lambda_1 < \lambda_2$, then $E(\lambda_1)^- \subset E_h(\lambda_1)^- \subset E(\lambda_2)$; and since $E_h(\lambda_1)^-$ is closed-and-open, it follows that $E(\lambda_1)$ is strongly contained in $E(\lambda_2)$. Theorem 6 now shows that $E(\lambda)$ is the spectral family of a continuous function f, $E(\lambda) = E_f(\lambda)$. Since $E_{f_\rho}(\lambda) \subset E_h(\lambda) = \bigcup_{\mu<\lambda} E_h(\mu) \subset E(\lambda)$ for all ρ, it follows that $f \leqq f_\rho$ for every ρ, in accordance with Theorem 5. On the other hand, if g is a continuous function such that $g \leqq f_\rho$ for all ρ, we have $E_g(\lambda) \supset E_{f_\rho}(\lambda)$ for all ρ and hence $E_g(\lambda) \supset \bigcup_\rho E_{f_\rho}(\lambda) \supset E_h(\lambda)$. Since g is continuous, it is true that $E_g(\lambda) \supset E_g(\mu)^-$ when $\lambda > \mu$. Hence $E_g(\lambda) \supset E_g(\mu)^- \supset E_h(\mu)^-$ for $\lambda > \mu$ and $E_g(\lambda) \supset \bigcup_{\mu<\lambda} E_h(\mu)^- = E_f(\lambda)$. Theorem 5 then shows that $g \leqq f$. Hence the function f is the greatest lower bound in $C(X)$ for the family of functions f_ρ. This completes the proof that the property $(B_{\aleph})$ is true for X and $C(X)$. The property $(A_{\aleph})$ is, of course, equivalent to $(B_{\aleph})$.

When the cardinal number of $C(X)$ does not exceed $\aleph$ the results obtained above may be expressed in the following terms:

THEOREM 14. *For a completely regular space X, the conditions (A) and (B) hold if and only if every open set has a closure which is open. For a topological space X, the latter condition implies both (A) and (B).*

Proof. If X is a completely regular space for which (A) or (B) is valid, then Theorem 12 shows that every open set is the union of closed-and-open sets and, as such, has a closure which is an open set. When X is a topological space in which every open set has a closure which is open, the condition (1) of Theorem 13 is obviously verified. As for condition (2), we observe that, if f is a continuous function and $\lambda > \mu$, then $E_f(\mu) \subset E_f(\mu)^- \subset E_f(\lambda)$ and $E_f(\lambda) = \bigcup_{\mu<\lambda} E_f(\mu)^-$ where the set $E_f(\mu)^-$ is open as well as closed.

There is no corresponding simplification in the other extreme case, $\aleph = \aleph_0$, unless X is assumed to be a normal space. We can then assert:

THEOREM 15. *In order that a normal Hausdorff space X have the properties $(A_{\aleph_0})$, $(B_{\aleph_0})$ it is necessary that every open F_σ-set in X have a closure which is open; and in order that a normal space X have the properties $(A_{\aleph_0})$, $(B_{\aleph_0})$ it is sufficient that the latter condition hold.*

Proof. It is well known that in a normal space an open set is an F_σ-set if and only if it is a spectral set $E_f(\lambda)$ for some continuous function f. Hence in a normal Hausdorff space with the properties $(A_{\aleph_0})$, $(B_{\aleph_0})$ we see by Theorem 11 (1) that every open F_σ-set has a closure which is open. In a normal space with the latter property, the condition (2) follows from the fact that $E_f(\lambda) = \bigcup_{n=1}^{\infty} E_f(\lambda - 1/n)^-$ where $E_f(\lambda - 1/n)^-$ is open as well as closed.

Theorems 14 and 15 have been given by H. Nakano in a paper published subsequently to our original announcement.[3]

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[3] H. Nakano, *Proc. Imp. Acad. Tokyo*, vol. 17 (1941), 308-310.

3. The case of a compact Hausdorff space. If we now concentrate our attention upon the case of a compact Hausdorff space X, we can simplify the results developed in Sec. **2**. The simplification results from the following facts.

THEOREM 16. *If X is a compact Hausdorff space, then*

(1) *X is completely regular;*

(2) *no non-void open set is of the first category;*

(3) *if f is a continuous function, then the spectral set $E_f(\lambda)$ is the union of at most $\aleph_0$ sets chosen from any specified additive basis for the open sets in X.*

Proof. (1) is well known, and (2) follows by application of the sufficient condition given at the time when (2) was introduced as hypothesis (C). To prove (3) we note first that $E_f(\lambda)$ is an F_σ-set, being the union of the closed sets $F_n = \{x; f(x) \leqq \lambda - 1/n\}$. From any specified additive basis for the open sets of X, let us select a family of open sets having $E_f(\lambda)$ as their union. This family is a covering for F_n and therefore contains a finite subcovering for F_n, by virtue of the compactness of X. We thus obtain a sequence of such finite coverings. Taken together they clearly provide a countable exact covering of $E_f(\lambda)$ by sets from the specified basis.

In view of Theorem 14 and the results of Sec. **2**, we now have the following result.

THEOREM 17. *In order that a compact Hausdorff space X have the equivalent properties $(A_\aleph)$, $(B_\aleph)$ it is necessary and sufficient that*

(1) *the closed-and-open sets constitute a basis for X;*

(2) *the closed-and-open sets have the $\aleph$-additivity property of Theorem 12.*

On any such space every bounded Baire function differs from a continuous function only on a set of the first category.

If we take into account the known results of the theory of Boolean algebras,[4] we can put the above result in the following form.

THEOREM 18. *A compact Hausdorff space has the equivalent properties $(A_\aleph)$, $(B_\aleph)$ if and only if it is the representative Boolean space for an $\aleph$-additive Boolean algebra.*

University of Chicago

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[4] See Stone, *Bull. Amer. Math. Soc.*, vol. 44 (1938), 807-816; *Trans. Amer. Math. Soc.*, vol. 40 (1936), 37-111, and vol. 41 (1937), 375-481. In Sec. 7 of the Bulletin paper (which is a brief general survey) the union of any non-void subclass of a Boolean ring is defined. The property of $\aleph$-additivity is the property that the union of every subclass of at most $\aleph$ members exists. In the topological representation (Theorem 67 of the first Transactions paper, and Theorems 1, 2, 4 of the second) this property is equivalent to the following: the union G of any family of at most $\aleph$ closed-and-open sets is contained in a smallest closed-and-open set, necessarily the closure of G.

SUBGROUPS OF FINITE INDEX IN FREE GROUPS

MARSHALL HALL, Jr.

1. Introduction. This paper[1] has as its chief aim the establishment of two formulae associated with subgroups of finite index in free groups. The first of these (Theorem 3.1) gives an expression for the total length of the free generators of a subgroup U of the free group F_r with r generators. The second (Theorem 5.2) gives a recursion formula for calculating the number of distinct subgroups of index n in F_r.

Of some independent interest are two theorems used which do not involve any finiteness conditions. These are concerned with ways of determining a subgroup U of F. The first (Theorem 4.1) gives a criterion for recognizing different representations of the same group, and the second (Theorem 5.1) yields a determination of a subgroup U by a set of permutations.

2. The standard representation. We shall be considering a free group F_r with a finite number of generators $s_1, s_2, \ldots, s_r$. A *word* f in F_r is any finite string $f = a_1 \ldots a_t$, each $a_i =$ some s_j or s_j^{-1} and the length of f, $l^*(f) = t$ the number of terms in the string. Every *element* f of F_r may be written in various ways as a word but in only one way as a reduced word. We shall write $l(f)$ for the length of the reduced form of the element f, whence

$$l^*(f) = l(f)$$

if and only if f is in reduced form.

A *Schreier system* is a set S of reduced words in F_r such that if any $f = a_1 \ldots a_t$ belongs to F_r then also $a_1 \ldots a_{t-1}$ belongs to F_r. If U is any subgroup of F_r then in the left cosets of U in F_r

$$F_r = Ug_1 + Ug_2 + \ldots + Ug_n \tag{2.1}$$

the representatives $g_1 = 1, g_2, \ldots, g_n$ may be chosen as a Schreier system. In this paper we shall be considering primarily subgroups U of finite index in F_r, $[F_r : U] = n$. In a recent paper [1] a standard representation for U was given. If G is a generic term for the g_i and S for the s_i, let $\phi(GS^\epsilon)$, $\epsilon = \pm 1$ be the G for the coset to which GS^ϵ belongs, H being a generic term for elements GS^ϵ, then ϕ is a function defined for arguments H. It was shown that U is completely determined by the Schreier system $S = \{G\}$ and the function $\phi(H)$. $\phi(H)$ satisfies:

(2.2.1) $\phi(H)$ is a G;

(2.2.2) $\phi(H) = G$ if $H = G$;

(2.2.3) $\phi[\phi(GS^\epsilon)S^{-\epsilon}] = G$.

Received March 2, 1948.

[1]This paper was written as part of a research project for the Office of Naval Research, under the administration of the Ohio State University Research Foundation.

Hence we may consider $\{G\}$ and $\phi(H)$ as giving a representation of U

$$U = U[\{G\}, \phi(H)] \tag{2.3}$$

which we shall call the standard representation for U.

3. Coset representatives and generators. Suppose the subgroup U of F_r to be given by its standard representation (2.3). Then an element u

$$u = u(i, a) = g_i s_a \phi(g_i s_a)^{-1} \tag{3.1}$$

is either a reduced word as it stands or reduces to the identity [1, §3]. Moreover, those u's different from the identity are a free set of generators of U. Since both g_i and $\phi(g_i s_a)^{-1} = g_j^{-1}$ are reduced as they stand, $u(i, a)$ reduces to the identity if and only if one of two things happens: (1) g_i ends in s_a^{-1}, (2) g_j ends in s_a.

Let $n(s_a)$ be the number of G's ending in s_a, $n(s_a^{-1})$ the number ending in s_a^{-1}. Then

$$\sum_a [n(s_a) + n(s_a^{-1})] = n - 1,$$

since every $G \neq 1$ is counted exactly once on the left. Hence, as Schreier proved [2] the number of generators of U is $nr - (n-1) = n(r-1) + 1 = N$. But there is also a relation connecting the lengths of the generators. Let us write

$$L = \sum_{i=1}^{n} l(g_i), \tag{3.2.1}$$

$$K = \sum_{k=1}^{N} l(u_k), \tag{3.2.2}$$

where in (3.2.2) $u_1, \ldots, u_N$ are the generators $u(i, a) \neq 1$ of U. From 3.1 it follows that

$$\sum_{i=1}^{n} l^*[u(i,a)] = \sum_{i=1}^{n} l^*[g_i s_a \phi(g_i s_a)^{-1}] = 2L + n, \tag{3.3}$$

since the $g_j = \phi(g_i s_a)$ are a permutation of the G's. In (3.3) let us separate the summation into three classes: (A) terms for which g_i ends in s_a^{-1}; (B) terms for which $g_j = \phi(g_i s_a)$ ends in s_a and (C) the remaining terms. Then

$$2L + n = \sum_A l^*[u(i, a)] + \sum_B l^*[u(i, a)] + \sum_C l^*[u(i, a)]. \tag{3.4}$$

If we now write

$$\lambda(s_a) = \sum_j l(g_j), \ g_j \text{ ends in } s_a; \tag{3.5}$$

$$\lambda(s_a^{-1}) = \sum_i l(g_i), \ g_i \text{ ends in } s_a^{-1};$$

we have

$$2L + n = 2\lambda(s_a^{-1}) + 2\lambda(s_a) + \sum_C l[u(i,a)], \tag{3.6}$$

since for $\sum_A$ if g_i ends in s_a^{-1}, $l[\phi(g_i s_a)] = l(g_i) - 1$ and $l^*[g_i s_a \phi(g_i s_a)^{-1}] = 2l(g_i)$. Similarly for $\sum_B$ if $g_j = \phi(g_i s_a)$ ends in s_a, then $l[\phi(g_j)] = l(g_i) + 1$ and $l^*[g_i s_a \phi(g_i s_a)^{-1}] = 2l(g_j)$. Summing (3.6) over $a = 1, 2, \ldots, r$ we have

$$(3.7) \qquad (2L + n)r = 2L + \sum_{k=1}^{N} l(u_k),$$

since $\sum_a [\lambda(s_a^{-1}) + \lambda(s_a)] = L$ and the u's of class C for $a = 1, 2, \ldots, r$ are precisely the $N = (r-1)n + 1$ free generators of U. Hence,

THEOREM 3.1. *Let U be a subgroup of index n in a free group F_r with r generators, given by its standard representation. Thus, if L is the total length of the coset representatives and K the total length of the free generators of U given by the standard representation, then $K = (2L + n)r - 2L$.*

4. Different representations of the same subgroup. Given a subgroup U of F, the choice of coset representatives, even as a Schreier system, is not in general unique. The choice of representatives, given U, determines the function ϕ. We seek a criterion for recognizing different representations of the same subgroup. The following theorem does not assume either that the number of generators of F is finite or that U is of finite index.

THEOREM 4.1. *Let $U_1 = U_1[\{G^1\}, \phi^1(H)]$, $U_2 = U_2[\{G^2\}, \phi^2(H)]$ be standard representations of subgroups U_1 and U_2 of the free group F. Then $U_1 = U_2$ if and only if we may find a one to one correspondence between the coset representatives mapping the identity onto itself such that if $g_i^1 \rightleftarrows g_j^2$, including $1 = g_1^1 \rightleftarrows g_1^2 = 1$ then for any s_a^ϵ, $\phi^1(g_i^1 s_a^\epsilon) \rightleftarrows \phi^2(g_j^2 s_a^\epsilon)$.*

Proof. First suppose $U_1 = U_2 = U$. Then if $Ug_i^1 = Ug_j^2$ is the same left coset of U as given in the two representations this establishes a $1-1$ correspondence $g_i^1 \rightleftarrows g_j^2$ which includes $1 = g_1^1 \rightleftarrows g_1^2 = 1$ such that if $g_i^1 \rightleftarrows g_j^2$ then $\phi^1(g_i^1 s_a^\epsilon) \rightleftarrows \phi^2(g_j^2 s_a^\epsilon)$. Conversely, suppose the $1-1$ correspondence given with $1 = g_1^1 \rightleftarrows g_1^2 = 1$ such that if $g_i^1 \rightleftarrows g_j^2$ then $\phi^1(g_i^1 s_a^\epsilon) \rightleftarrows \phi^2(g_j^2 s_a^\epsilon)$. By induction on length we may show that an element belonging to a coset of U_1 belongs to the corresponding coset of U_2. This is true for $l(f) = 0$, since $f = 1$, and $1 \rightleftarrows 1$. And if f is in the cosets $U_1 g_i^1 \rightleftarrows U_2 g_j^2$, then fs_a^ϵ is in the corresponding cosets $U_1\phi^1(g_i^1 s_a^\epsilon) \rightleftarrows U_2\phi^2(g_j^2 s_a^\epsilon)$. Hence corresponding cosets contain the same elements and in particular $U_1 = U_2$. Examples exist (subgroups of index 3 in F_2) showing that the requirement $1 = g_1^1 \rightleftarrows g_1^2 = 1$ is not redundant. As an illustration of the content of Theorem 3.1, it may be observed that there are 18 Schreier systems for subgroups of index 3 in F_2 and each of these yields four or six subgroups of index 3, and yet of these only 13 are different.

5. The number of subgroups of index n in F_r. If U is a subgroup of any group F (free or not) generated by $s_1, \ldots, s_i, \ldots$ then by multiplication on the right each s_i induces a permutation P_i on the left cosets of U in F. Since the s_i's generate F, the P_i's will generate a group transitive on the cosets. The following theorem is a kind of converse for free groups. No finiteness assumptions are needed here.

THEOREM 5.1. *Given a free group F with generators $s_1, \ldots s_i, \ldots$ and a set of indices $I = \{1, \ldots, t, \ldots\}$. With each generator s_i associate a permutation P_i of the indices. Suppose $J = \{1, \ldots, j, \ldots\}$ to be the subset of I which is*

the transitive constituent of I including the index 1. Then in F there is a Schreier system $\{G\} : g_1 = 1, \ldots, g_j, \ldots$, which may be indexed by J and a ϕ function such that $\phi(g_j s_i^r) = g_k$ if and only if P_i^r takes j into k.

Proof. The permutations P_i generate a group E. In E let D_1 be the subgroup which fixes the symbol 1. The mapping $s_i \to P_i$ determines a homomorphism of F onto E, $F \to E$. In this homomorphism let U be the subgroup of F mapped onto D_1, $U \to D_1$. We may choose coset representatives $\{G\}$ of U in F as a Schreier system. If $G \to A$, then $UG \to D_1 A$. Let A take the index 1 into j, which we write $(1)A = j$. Here let us assign the index j to G, putting $G = g_j$. Hence the Schreier system $\{G\}$ has been indexed by those indices into which elements of E take the index 1, which form the set J. Here if $g_j \to A$, $s_i^r \to P_i^r$, then $g_j s_i^r \to A P_i^r$. Now if $(j)P_i^r = k$, then $(1)AP_i^r = k$ and AP_i^r belongs to the coset $D_1 B$ of D_1, consisting of those elements X of E such that $(1)X = k$. Here $Ug_k \to D_1 B$. Hence $g_j s_i^r$ belongs to Ug_k or $\phi(g_j s_i^r) = g_k$.

THEOREM 5.2. *The number $N_{n,r}$ of subgroups of index n in F_r is given recursively by $N_{1,r} = 1$,*

$$N_{n,r} = n(n!)^{r-1} - \sum_{i=1}^{n-1} [(n-i)!]^{r-1} N_{i,r}.$$

Proof. $N_{1,r} = 1$ states merely that F_r is its own unique subgroup of index 1. From Theorem 5.1 each subgroup U of index a is determined by r permutations $P_1, \ldots, P_r$, generating a group transitive on numbers $1, b_2, \ldots, b_a$. From Theorem 4.1 replacement of $1, b_2, \ldots, b_a$ by $1, c_2, \ldots, c_a$ in $P_1, \ldots, P_r$ yields the same group U.

Consider r permutations $P_1, P_2, \ldots, P_r$ on n letters. The number of possible choices for $P_1, P_2, \ldots, P_r$ is $(n!)^r$. In general $P_1, \ldots, P_r$ will not generate a group transitive on all n letters. Let $1, b_2, b_3, \ldots, b_a$ be the transitive constituent which includes the identity. Disregarding the rest of the numbers, we may associate $P_1, \ldots, P_r$ with a unique subgroup of index a. The other $n - a$ letters may occur in $[(n-a)!]^r$ ways. Also $b_2, \ldots, b_a$ could be replaced by numbers $c_2, \ldots, c_a$ from $2, \ldots, n$ in $(n-1)(n-2) \ldots (n-a+1)$ ways. Hence, of the permutations $P_1, \ldots, P_r$, on $1, 2, \ldots, n$ a total of $(n-1)(n-2) \ldots (n-a+1)[(n-a)!]^r = (n-1)![(n-a)!]^{r-1}$ are associated with the same subgroup of index a in F_r. Hence $(n-1)![(n-a)!]^{r-1} N_{a,r}$ permutations are associated with subgroups of index a. Hence

$$(n!)^r = \sum_{a=1}^{n} (n-1)![(n-a)!]^{r-1} N_{a,r}.$$

If here we divide by $(n-1)!$ and transpose the sum for $a = 1$ to $n-1$ we have the formula of the Theorem.

Ohio State University

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ON THE UNIQUENESS OF THE GREEN'S FUNCTION ASSOCIATED WITH A SECOND-ORDER DIFFERENTIAL EQUATION

E. C. TITCHMARSH

SUMMARY

The Green's function $G(x, \xi, \lambda)$ associated with the differential equation

$$\frac{d^2\phi}{dx^2} + \{\lambda - q(x)\}\,\phi = 0 \qquad (x \geqq 0)$$

is of importance in the theory of the expansion of an arbitrary function in terms of the solutions of the differential equation. It is proved that this function is unique if $q(x) \geqq -Ax^2 - B$, where A and B are positive constants or zero. A similar theorem is proved for the Green's function $G(x, y, \xi, \eta, \lambda)$ associated with the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \{\lambda - q(x, y)\}\,\phi = 0.$$

1. The problem of expanding an arbitrary function in terms of solutions of the differential equation

$$\frac{d^2\phi}{dx^2} + \{\lambda - q(x)\}\,\phi = 0 \tag{1.1}$$

depends on the properties of a function $G(x, \xi, \lambda)$ called the Green's function.[1] We shall consider here the case in which the interval of values of x is $(0, \infty)$, and where the solutions $\phi(x, \lambda)$ of (1.1) to be used in the expansion satisfy the boundary condition

$$\phi(0, \lambda)\cos\alpha + \phi_x(0, \lambda)\sin\alpha = 0, \tag{1.2}$$

suffixes denoting partial differentiations. In this case the Green's function has the following properties. Let ξ be a given positive number, λ a complex number whose imaginary part is not zero. Then if $x \neq \xi$, $G(x, \xi, \lambda)$ has continuous partial derivatives with respect to x up to the second order, and

$$\frac{\partial^2 G}{\partial x^2} + \{\lambda - q(x)\}\,G = 0. \tag{1.3}$$

Also

$$G(0, \xi, \lambda)\cos\alpha + G_x(0, \xi, \lambda)\sin\alpha = 0\,; \tag{1.4}$$

Received September 1, 1948.

[1] See D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Zweiter Abschnitt (Berlin, 1924); R. Courant and D. Hilbert, *Methoden der math. Physik* I, (Berlin, 1931), V, § 14; E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations* (Oxford, 1946), p. 29; P. Hartman and A. Wintner, "A Criterion for the Non-degeneracy of the Wave Equation," *Amer. J. Math.*, vol. 71 (1949), 206-213.

$$\int_0^\infty |G(x, \xi, \lambda)|^2 \, dx \tag{1.5}$$

is convergent; and $G(x, \xi, \lambda)$ is continuous at $x = \xi$, while[2]

$$G_x(\xi + 0, \xi, \lambda) - G_x(\xi - 0, \xi, \lambda) = -1. \tag{1.6}$$

The Green's function is also symmetrical in x and ξ.

If $\phi(x, \lambda)$ is a solution of (1.1) satisfying (1.2), and $\psi(x, \lambda)$ is a solution of integrable square over $(0, \infty)$, and the Wronskian of ϕ and ψ is 1, then the above conditions are satisfied by the function

$$\begin{aligned} G(x, \xi, \lambda) &= -\phi(x, \lambda)\psi(\xi, \lambda) && (x \leqq \xi), \\ &\quad -\psi(x, \lambda)\phi(\xi, \lambda) && (x > \xi). \end{aligned}$$

It was shown by Weyl[3] that there are two different cases. In one case, called the limit-point case, there is only one $\psi(x, \lambda)$ with the required properties for each λ, and the Green's function is then unique. In the other case, called the limit-circle case, all solutions of (1.1) are of integrable square over $(0, \infty)$, and the Green's function depends on an arbitrary parameter.

The object of the present paper is to consider how the uniqueness of the Green's function depends on the nature of $q(x)$. I consider also the corresponding problem for functions of two variables, in which (1.1) is replaced by

$$\nabla^2\phi + \{\lambda - q(x, y)\}\phi = 0, \tag{1.7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The extension to more than two variables seems to be immediate. No doubt more general differential equations could also be dealt with in a similar way.

2. For functions of one variable, we shall prove

THEOREM 1. *If $q(x) \geqq -Ax^2 - B$, where A and B are positive constants (or zero), then the Green's function is unique.*

Thus in particular the result holds if $q(x)$ is bounded below.

Suppose that two functions $G_1(x, \xi, \lambda)$ and $G_2(x, \xi, \lambda)$ satisfy the above conditions, and let

$$g(x, \lambda) = G_1(x, \xi, \lambda) - G_2(x, \xi, \lambda),$$

ξ now being fixed. Then $g(x, \lambda)$ also satisfies the same conditions, except that $g_x(x, \lambda)$ is continuous at $x = \xi$. Hence

$$\begin{aligned} &\int_0^X \{g(x, \lambda)g_{xx}(x, \mu) - g(x, \mu)g_{xx}(x, \lambda)\} \, dx \\ &= [g(x, \lambda)g_x(x, \mu) - g(x, \mu)g_x(x, \lambda)]_0^X \\ &= g(X, \lambda)g_x(X, \mu) - g(X, \mu)g_x(X, \lambda). \end{aligned}$$

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[3]Different constant factors are attached to the Green's function by different writers. The definition adopted here is that of Courant and Hilbert. The function in my book had the opposite sign.

[2]See chap. II of my book.

The left-hand side is also equal to

$$\int_0^X [g(x,\lambda)\{q(x)-\mu\}g(x,\mu) - g(x,\mu)\{q(x)-\lambda\}g(x,\lambda)]\,dx$$
$$= (\lambda-\mu)\int_0^X g(x,\lambda)g(x,\mu)\,dx.$$

Taking $\mu = \bar{\lambda}$ (the conjugate of λ), and writing $g(x,\lambda) = g(x)$, $g_x(x,\lambda) = g'(x)$, $g(x,\bar{\lambda}) = \bar{g}(x)$, we obtain

$$\Im(\lambda)\int_0^X |g(x)|^2\,dx = \Im\{g(X)\bar{g}'(X)\}. \tag{2.1}$$

If the right-hand side tended to zero as $X \to \infty$, we should have

$$\int_0^\infty |g(x)|^2\,dx = 0.$$

Hence we should have $g(x) = 0$ for all values of x, and the uniqueness of the Green's function would follow.

To prove the theorem stated, integrate (2.1) with respect to X over $(0, T)$ and divide by $T\Im(\lambda)$. We obtain

$$\int_0^T \left(1-\frac{x}{T}\right)|g(x)|^2\,dx = \frac{1}{T\Im(\lambda)}\int_0^T \Im\{g(x)\bar{g}'(x)\}\,dx$$
$$\leqq \frac{1}{T|\Im(\lambda)|}\int_0^T |g(x)g'(x)|\,dx$$
$$\leqq \frac{1}{T|\Im(\lambda)|}\left\{\int_0^T |g(x)|^2\,dx\int_0^T |g'(x)|^2\,dx\right\}^{\frac{1}{2}}$$
$$= O\left[\frac{1}{T}\left\{\int_0^T |g'(x)|^2\,dx\right\}^{\frac{1}{2}}\right] \tag{2.2}$$

as $T \to \infty$, λ remaining fixed, since $g(x)$ is of integrable square over $(0, \infty)$. The result will now follow if

$$\int_0^T |g'(x)|^2\,dx = o(T^2) \tag{2.3}$$

as $T \to \infty$, at least through some sequence of values; for then

$$\int_0^{\frac{1}{2}T} |g(x)|^2\,dx \leqq 2\int_0^T \left(1-\frac{x}{T}\right)|g(x)|^2\,dx \to 0,$$

and hence $g(x) = 0$ as before.

Now

$$\int_0^T |g'(x)|^2\,dx \leqq 2\int_0^{2T}\left(1-\frac{x}{2T}\right)|g'(x)|^2\,dx,$$

and

$$\int_0^{2T}\left(1-\frac{x}{2T}\right)|g'(x)|^2\,dx = \int_0^{2T}\left(1-\frac{x}{2T}\right)g'(x)\bar{g}'(x)\,dx$$

$$= \left[\left(1-\frac{x}{2T}\right) g(x)\bar{g}'(x)\right]_0^{2T} - \int_0^{2T}\left(1-\frac{x}{2T}\right) g(x)\bar{g}''(x)\,dx$$
$$+\frac{1}{2T}\int_0^{2T} g(x)\bar{g}'(x)\,dx$$
$$= -g(0)\bar{g}'(0) - \int_0^{2T}\left(1-\frac{x}{2T}\right) g(x)\left\{q(x)-\bar{\lambda}\right\}\bar{g}(x)\,dx$$
$$+\frac{1}{2T}\int_0^{2T} g(x)\bar{g}'(x)\,dx.$$

Taking real parts, we obtain

$$\int_0^{2T}\left(1-\frac{x}{2T}\right)|g'(x)|^2\,dx = -\,\Re\left\{g(0)\bar{g}'(0)\right\}$$
$$-\int_0^{2T}\left(1-\frac{x}{2T}\right)|g(x)|^2\left\{q(x)-\Re(\lambda)\right\}dx + \frac{1}{4T}\int_0^{2T}\left\{g(x)\bar{g}'(x)+\bar{g}(x)g'(x)\right\}dx$$
$$= -\,\Re\left\{g(0)\bar{g}'(0)\right\} - \int_0^{2T}\left(1-\frac{x}{2T}\right)|g(x)|^2\left\{q(x)-\Re(\lambda)\right\}dx + \frac{|g(2T)|^2-|g(0)|^2}{4T}.$$

Hence

$$\int_0^T |g'(x)|^2\,dx \leqq -\,2\Re\left\{g(0)\bar{g}'(0)\right\}$$
$$+2\int_0^{2T}|g(x)|^2\left\{Ax^2+B+|\Re(\lambda)|\right\}dx + \frac{|g(2T)|^2}{2T}.$$

Writing the integral on the right-hand side as

$$\int_0^{\sqrt{T}} + \int_{\sqrt{T}}^{2T}$$

it is seen to be of the form

$$O\left\{T\int_0^{\sqrt{T}}|g(x)|^2\,dx\right\} + O\left\{T^2\int_{\sqrt{T}}^{2T}|g(x)|^2\,dx\right\} = o(T^2).$$

Finally

$$|g(2T)| < K$$

for some K and some arbitrarily large values of T, since otherwise $g(x)$ could not be of integrable square. Hence (2.3) holds for some arbitrarily large values of T, and the uniqueness result follows.

3. It is not clear whether the above result is the best possible of its kind. But it is not far from being so, since the result fails if we replace the x^2 of the theorem by $x^{2+\delta}$, where δ is any positive number. This follows from the analysis of § 5.8 of my book, *Eigenfunction Expansions*, where it is shown that the limit-circle case holds if $q(x) \leqq 0$, $q'(x) < 0$, $q(x) \to -\infty$, $q'(x) = O\{|q(x)|^c\}$, $0 < c < \frac{3}{2}$, $q''(x)$ is ultimately of one sign, and

$$\int^{\infty} |q(x)|^{-\frac{1}{2}}\,dx$$

is convergent. These conditions are satisfied if $q(x) = -\,x^{2+\delta} - x$.

4. We shall next consider the corresponding problem for functions of two variables, the region considered being the whole (x, y) plane. In this case the Green's function $G(x, y, \xi, \eta, \lambda)$ satisfies the following conditions. Let ξ and η be given real numbers, λ a complex number whose imaginary part does not vanish. Then G has continuous partial derivatives up to the second order with respect to x and y, and

$$\nabla^2 G + \{\lambda - q(x, y)\} G = 0, \tag{4.1}$$

except at $x = \xi, y = \eta$. The integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x, y, \xi, \eta, \lambda)|^2 \, dxdy \tag{4.2}$$

is convergent. In the neighbourhood of $x = \xi, y = \eta$,

$$G(x, y, \xi, \eta, \lambda) = \frac{1}{2\pi} \log \frac{1}{\rho} + O(1), \tag{4.3}$$

$$\frac{\partial G}{\partial \rho} = -\frac{1}{2\pi\rho} + O(1), \tag{4.4}$$

where

$$\rho = \sqrt{\{(x - \xi)^2 + (y - \eta)^2\}}.$$

It can be shown that such a function exists, at any rate if $q(x, y)$ is continuous and has continuous partial derivatives of the first order.[4]

THEOREM 2. *If $q(x, y) \geqq -Ar^2 - B$, where $r = \sqrt{(x^2 + y^2)}$, and A and B are positive constants (or zero), then the Green's function is unique.*

Suppose that two functions $G_1(x, y, \xi, \eta, \lambda)$ and $G_2(x, y, \xi, \eta, \lambda)$ satisfy the above conditions, and let

$$g(x, y, \lambda) = G_1(x, y, \xi, \eta, \lambda) - G_2(x, y, \xi, \eta, \lambda),$$

ξ and η now being fixed. Then $g(x, y, \lambda)$ satisfies the same conditions, except that g and $\partial g/\partial \rho$ are bounded as $\rho \to 0$.

If r, θ, are the polar coordinates corresponding to x, y, we write

$$g(x, y, \lambda) = g[r, \theta, \lambda].$$

Then[5]

$$\iint_{r \leqq R} \{g(x, y, \lambda)\nabla^2 g(x, y, \mu) - g(x, y, \mu)\, \nabla^2 g(x, y, \lambda)\} \, dxdy$$

$$= \int_0^{2\pi} \left\{ g[R, \theta, \lambda] \frac{\partial}{\partial R} g[R, \theta, \mu] - g[R, \theta, \mu] \frac{\partial}{\partial R} g[R, \theta, \lambda] \right\} R d\theta .$$

The left-hand side is also equal to

$$\iint_{r \leqq R} [g(x, y, \lambda) \{q(x, y) - \mu\} g(x, y, \mu) - g(x, y, \mu) \{q(x, y) - \lambda\} g(x, y, \lambda)] \, dxdy$$

$$= (\lambda - \mu) \iint_{r \leqq R} g(x, y, \lambda)\, g(x, y, \mu) \, dxdy .$$

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[4] See a forthcoming paper in the *Proc. London Math. Soc.*

[5] See e.g. E. Goursat, *Cours d'Analyse mathématique*, vol. 3 (Paris, 1927), § 506.

Taking $\mu = \bar{\lambda}$, we obtain

$$\Im(\lambda) \iint_{r \leqq R} |g(x, y, \lambda)|^2 \, dxdy = \int_0^{2\pi} \Im\left\{ g[R, \theta, \lambda] \frac{\partial}{\partial R} g[R, \theta, \bar{\lambda}] \right\} R d\theta .$$

Now integrate with respect to R over $(0, T)$ and divide by $T\Im(\lambda)$. Writing $g[r, \theta, \lambda] = g$, $\frac{\partial g}{\partial r} = g_r$, etc., we have

$$\iint_{r \leqq T} \left(1 - \frac{r}{T}\right) |g|^2 r dr d\theta = \frac{1}{T\Im(\lambda)} \int_0^T \int_0^{2\pi} \Im(g\bar{g}_r) \, r dr d\theta$$

$$\leqq \frac{1}{T|\Im(\lambda)|} \left\{ \int_0^T \int_0^{2\pi} |g|^2 r dr d\theta \int_0^T \int_0^{2\pi} |g_r|^2 \, r dr d\theta \right\}^{\frac{1}{2}}$$

$$= O\left[\frac{1}{T} \left\{ \int_0^T \int_0^{2\pi} |g_r|^2 \, r dr d\theta \right\}^{\frac{1}{2}} \right],$$

as $T \to \infty$. It is therefore sufficient to prove that

$$\int_0^T \int_0^{2\pi} |g_r|^2 \, r dr d\theta = o(T^2) \tag{4.5}$$

as $T \to \infty$ through some sequence of values; for then

$$\iint_{r \leqq 1} |g|^2 \, r dr d\theta \leqq 2 \iint_{r \leqq T} \left(1 - \frac{r}{T}\right) |g|^2 \, r dr d\theta \to 0,$$

and hence $g = 0$ for all values of r and θ.

Now

$$\int_0^T \int_0^{2\pi} |g_r|^2 \, r dr d\theta \leqq \int_0^T \int_0^{2\pi} \left(|g_r|^2 + \frac{1}{r^2} |g_\theta|^2 \right) r dr d\theta$$

$$\leqq 2 \int_0^{2T} \left(1 - \frac{r}{2T}\right) r dr \int_0^{2\pi} \left(g_r \bar{g}_r + \frac{1}{r^2} g_\theta \bar{g}_\theta \right) d\theta$$

$$= 2 \iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) (g_x \bar{g}_x + g_y \bar{g}_y) \, dxdy .$$

By another known formula[6]

$$\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)g\nabla^2\bar{g}\,dxdy$$

$$=-\iint_{r\leqq 2T}\left[\frac{\partial}{\partial x}\left\{\left(1-\frac{r}{2T}\right)g\right\}\bar{g}_x+\frac{\partial}{\partial y}\left\{\left(1-\frac{r}{2T}\right)g\right\}\bar{g}_y\right]dxdy$$

$$=-\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)(g_x\bar{g}_x+g_y\bar{g}_y)\,dxdy$$

$$+\frac{1}{2T}\iint_{r\leqq 2T}\left(\frac{\partial r}{\partial x}\bar{g}_x+\frac{\partial r}{\partial y}\bar{g}_y\right)g\,dxdy$$

$$=-\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)(g_x\bar{g}_x+g_y\bar{g}_y)\,dxdy+\frac{1}{2T}\iint_{r\leqq 2T}g\bar{g}_r\,rdrd\theta,$$

since

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\frac{\partial x}{\partial r},$$

etc. Hence

$$\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)(g_x\bar{g}_x+g_y\bar{g}_y)\,dxdy$$

$$=-\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)g\nabla^2\bar{g}.rdrd\theta+\frac{1}{2T}\iint_{r\leqq 2T}g\bar{g}_r rdrd\theta$$

$$=-\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)g\,(q-\bar{\lambda})\,\bar{g}.rdrd\theta+\frac{1}{2T}\iint_{r\leqq 2T}g\bar{g}_r rdrd\theta.$$

Taking real parts, we obtain

$$\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)(g_x\bar{g}_x+g_y\bar{g}_y)\,dxdy$$

$$=-\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)|g|^2\{q-\Re(\lambda)\}\,rdrd\theta+\frac{1}{2T}\iint_{r\leqq 2T}(g\bar{g}_r+\bar{g}g_r)\,rdrd\theta.$$

Also

$$\int_{r\leqq 2T}(g\bar{g}_r+\bar{g}g_r)\,rdr=[|g|^2r]_0^{2T}-\int_0^{2T}|g|^2\,dr\leqq 2T\,|g\,[2T,\theta]|^2.$$

---

[6]It can be verified at once by integration by parts.

Altogether, we obtain

$$\int_0^T\int_0^{2\pi} |g_r|^2\, r dr d\theta \leqq -2\iint_{r\leqq 2T}\left(1-\frac{r}{2T}\right)|g|^2\{q-\Re(\lambda)\}\, r dr d\theta + 2\int_0^{2\pi}|g[2T,\theta]|^2\, d\theta.$$

Using the given condition, it follows that

$$\int_0^T\int_0^{2\pi} |g_r|^2\, r dr d\theta \leqq 2\iint_{r\leqq 2T}|g|^2\{Ar^2+B+|\Re(\lambda)|\}\, r dr d\theta + 2\int_0^{2\pi}|g[2T,\theta]|^2\, d\theta.$$

The first integral on the right is

$$O\left\{T\iint_{r\leqq\sqrt{T}}|g|^2\, r dr d\theta\right\} + O\left\{T^2\iint_{\sqrt{T}<r\leqq 2T}|g|^2\, r dr d\theta\right\} = o(T^2).$$

Also

$$\int_0^{2\pi}|g[2T,\theta]|^2\, d\theta < K$$

for some K and some arbitrarily large values of T, since otherwise

$$\int_0^R r dr\int_0^{2\pi}|g[r,\theta]|^2\, d\theta > K'R^2.$$

The theorem therefore follows as before.

It seems likely that this result also is not far from the best possible, but in the case of functions of two or more variables the different possibilities have not been explored in sufficient detail to say anything definite on this point.

Oxford, England

NOTE ON NEWTONIAN FORCE-FIELDS

GARRETT BIRKHOFF AND LINDLEY BURTON

1. Introduction.[1] Although the behaviour of Newtonian potentials *inside* n-dimensional distributions of mass or charge has been discussed in the sense of Lebesgue-Stieltjes integrals by various authors,[2] the discussion of various important theorems seems to have been made only in the sense of Riemann integration, and assuming the Hölder conditions[3] (or at least piecewise continuity) for the volume density ρ. We shall generalize these theorems below.

The discussion may be divided into three parts. First, we discuss Gauss' Integral Theorem. Then, we prove that the force components exist and are the derivatives of a potential, under very general conditions. In both cases, the Fubini Theorem is the principal tool.[4] Finally, we discuss Poisson's equation.

$$\nabla^2 U + (n-2)\omega_n \rho = 0, \quad [n \geq 3], \tag{1}$$

$$\nabla^2 U + 2\pi\rho = 0, \quad [n = 2]. \tag{1'}$$

Here ω_n is the surface area of the unit sphere in n-space. We also extend the results of H. Petrini in various ways.[5]

2. Gauss' Integral Theorem. Consider a charge distribution of variable density $\rho(x) = \rho(x_1, \ldots, x_n)$, zero outside a bounded region R of Euclidean n-space.[6] We define the *potential* at $a = (a_1, \ldots, a_n)$ as usual by

$$U(a) = \int_R \rho(x) dV / r^{n-2} \quad \text{if } n \geq 3, \tag{2}$$

and the attractive *force*-components by

Received May 4, 1948.

[1] The results of **2-3** were announced at the Princeton Bicentennial Conference in 1946; see Abstract 52-5-137 of *Bull. Amer. Math. Soc.*

[2] Notably Lichtenstein [2]; G. C. Evans, Rice Institute Pamphlet 7 (1920), 252-329; F. Riesz, *Acta Math.*, vol. 48 (1926), 329-343 and vol. 54 (1930), 321-360; T. Rado, *Subharmonic Functions* (Berlin, 1937). We are indebted to Professors Evans, Rado, and Zygmund for various comments with bibliographical suggestions.

[3] O. Hölder, *Beiträge zur Potentialtheorie* (Stuttgart, 1882), 10; Kellogg [1], 150-156; Courant-Hilbert, *Methoden der math. physik*, vol. 2, 228; J. Kravtchenko, *J. Math. Pures Appl.*, vol. 23 (1944), 97-210.

[4] It has been used before in potential theory. See G. C. Evans, *Trans. Amer. Math. Soc.*, vol. 37 (1935), 256; J. M. Thompson, *Bull. Amer. Math. Soc.*, vol. 41 (1935), 744-752; M. O. Reade, *ibid.*, vol. 53 (1947), 321-331.

[5] H. Petrini [3] proved that $\nabla^2 U$ was equal to $-4\pi\rho$ wherever ρ was continuous and $\nabla^2 U$ existed. He also showed that a "generalized Laplacian" existed wherever ρ was continuous. Lichtenstein [2] showed that if ρ was of class L_2, then Poisson's equation was valid except on a set of measure zero.

[6] We could easily represent ρ as the sum of a positive and negative distribution, and so restrict ourselves to potentials of positive mass.

$$(3) \qquad F_i(a) = (n-2)\int_R (x_i - a_i)\rho dV/r^n = \int_R [\partial/\partial a_i(1/r^{n-2})]\rho dV.$$

If $n = 2$, we define correspondingly

$$(2') \qquad U(a) = -\int_R \rho(x)\ln r dV,$$

$$(3') \qquad F_i(a) = \int_R (x_i - a_i)\rho dV/r^2 = -\int_R [\partial/\partial a_i(\ln r)]\rho dV.$$

Here $dV = dx_1 \ldots dx_n$ and $r = |x - a| = [\sum_i (x_i - a_i)^2]^{\frac{1}{2}}$. The integrals may be interpreted either in the sense of Lebesgue or of (improper) Riemann integration. For the Lebesgue integral of a function $f(x)$ to exist, $f(x)$ must be *measurable*; for the Riemann integral to exist, it must be[7] *continuous almost everywhere*—i.e., except on a set of Lebesgue measure zero. Since all factors except ρ in definitions (2), (3), (2'), (3') are continuous except at $x = a$, these conditions are satisfied by the integrands if they are satisfied by ρ. — For the integrals to exist it is necessary and sufficient that in addition the upper integral of $|f|$ be finite. Hence it is necessary that the total charge $C = \int_R |\rho| dV$ be finite, interpreted in the sense of Lebesgue integration. We recall also that if the Riemann integral exists, then so does the Lebesgue integral, and the two are equal.

Now let N denote the force-component in the direction of the outward normal to a (hyper) surface and let ω_n denote the hyper-area of the unit sphere in n-space. Further, let the concept of "regular" surface defined in [1] be extended in the natural way to n dimensions; let $\partial/\partial n$ denote the outward normal directional derivative. For any regular region R (i.e., region bounded by a regular surface S),

$$(4) \qquad \int_S |\partial(1/r^{n-2})/\partial n|\, ds = K \text{ is finite.}$$

Thus it is $(n-2)\omega_n$ for a convex region.

THEOREM 1. *If $\rho(x)$ is Lebesgue integrable over R then for any "regular" hypersurface bounding a region T,*

$$(5) \qquad \int_S N dS = (2-n)\omega_n \int_T \rho dV.$$

Proof. Form the $(2n-1)$-dimensional product space $S \times R$. Clearly $\rho(x)\partial(1/r^{n-2})/\partial n$ is measurable over $S \times R$, since the factors are measurable over R and S respectively. Further, by (4) and our earlier remarks,

$$\int_{S\times R} |\rho(x)\partial(1/r^{n-2})/\partial n| dS dV \leq K\int_R |\rho| dV \leq KC$$

is finite; hence the Fubini Theorem ([5], p. 87) applies. Taking this as an

[7] We could not locate an explicit statement of these results, for $n > 1$, in the literature.

iterated integral in the two possible orders, we get (5) precisely.—Note that we can disregard points on $S \times S$, which has $(2n-1)$-dimensional measure zero.

By the Fubini Theorem again, if the $\partial F_i/\partial x_i$ are all Lebesgue integrable over a coordinate cuboid (i.e., rectangle in n dimensions), C, the Divergence Theorem holds[8] for C. Hence, by (5), Div F and $(2-n)\omega_n\rho$ have the same integral over any coordinate cuboid C. From this it follows immediately (by the Lebesgue density theorem or otherwise) that the integrands are equal almost everywhere. This proves the following result.

COROLLARY 1. *If $\rho(x)$ and the $\partial F_i/\partial x_i$ exist almost everywhere and are Lebesgue integrable on a region R, then*

$$\text{Div } F = (2-n)\omega_n\rho, \text{ almost everywhere in } R. \tag{1}$$

A much more elementary argument of the same type shows that if ρ is continuous, and if $\nabla^2 U$ is also continuous, then (1) holds everywhere.

The argument of Theorem 1 gives another interesting result in case S is a sphere of radius c and centre O. It is well-known, and easy to show,[9] that the *average* $\overline{U}(S)$ on S of the potential due to a charge e at x is equal to the potential e/r^{n-2} at the centre, if x is outside S, and is e/c^{n-2} if x is inside S. Using the Fubini Theorem as before, we conclude

COROLLARY 2. *If $\rho(x)$ is Lebesgue integrable, then $U(a)$ exists almost everywhere on S, and*

$$\overline{U}(S) = \int_{r=c}^{\infty} \rho(x)dV/r^{n-2} + (1/c^{n-2})\int_{r=0}^{c} \rho dV. \tag{6}$$

Moreover if (2) exists as a Lebesgue integral when $a = O$

$$\overline{U}(S) - U(O) = \int_{r=0}^{c} (1/c^{n-2} - 1/r^{n-2})\rho dV. \tag{7}$$

A similar argument may be applied to solid spherical averages, which we denote $U^*(S)$. It gives, in three dimensions

$$U^*(S) - U(O) = \int_{r=0}^{c} \frac{1}{2}\left(\frac{3}{c} - \frac{r^2}{c^3} - \frac{2}{r}\right)\rho dV = \frac{-1}{2c^3}\int_{r=0}^{c} \frac{1}{r}(c-r)^2(2c+r)\zeta dV. \tag{8}$$

These differences can be used to define generalized Laplacians.[10]

3. Force and derivatives of potential. The assumptions involved in Corollary 1 are inconvenient, because they involve both ρ and the F_i. We shall now give some results involving only assumptions about ρ.

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[8] See H. Federer, "The Gauss-Green Theorem," *Trans. Amer. Math. Soc.*, vol. 58 (1945), 44-76, esp. pp. 48-50, 60-62; G. G. Lorentz, *Bericht über die Mathematiker-Tagung in Tübingen*, 23-7 Sept., 1946, pp. 94-6.

[9] The average potential on S due to a charge at x is equal by the symmetry of $1/r$ to the potential at x due to the same *total* charge uniformly distributed on S.

[10] Introduced by Evans and F. Riesz, *op. cit., supra.* See Courant-Hilbert, *op. cit., supra*, p. 258; and especially S. Saks, *Mat. Sbornik*, vol. 51 (1941), 451-456, and I. Privaloff, *ibid.*, 457-460.

THEOREM 2. *The $F_i(a)$ defined by* (3) *all exist if and only if ρ is measurable (continuous a.e.) and*

$$\int_R |\rho| dV/r^{n-1} < +\infty. \tag{9}$$

Explanation. In the case of Lebesgue integrals, ρ must be measurable; with improper Riemann integrals, it must be continuous a.e. In the latter case, (9) must be interpreted as meaning that the proper Riemann integrals of $|\rho| dV/r^{n-1}$ over $R - S(\delta)$, where $S(\delta)$ denotes a sphere of radius δ with centre a, have a finite upper bound. In any case, we note that (9) implies

$$g(\delta) = \int_{S(\delta)} |\rho| dV/r^{n-1} \to 0 \text{ as } \delta \to 0. \tag{9'}$$

Proof. As above, a measurable (continuous a.e.) function in $n > 1$ dimensions is Lebesgue integrable (respectively Riemann integrable) if and only if the upper integral of its absolute value is finite. But

$$\left| \rho \frac{x_i - a_i}{r^n} \right| = \frac{|\rho|}{r^{n-1}} \cdot \frac{|x_i - a_i|}{r} \leqq \frac{|\rho|}{r^{n-1}} \leqq \sum_{i=1}^{n} \left| \rho \frac{x_i - a_i}{r^n} \right|.$$

Hence the integral (9) is finite if and only if all the force integrals in (3) are finite—i.e., exist.

COROLLARY 1. *The theorem holds if* (9) *is replaced by*

$$\rho = 0\left(\frac{1}{r^{1-\gamma}}\right) \text{ or even } \rho = 0\left(\frac{1}{r\,|\log r|^{1+\gamma}}\right)$$

in the neighbourhood of the origin. Here $\gamma > 0$.

However, the existence of the force integrals (3) does not imply that the potential integral (2) is differentiable. We now investigate sufficient conditions for this.

THEOREM 3. *In order that the two terms of*

$$F_i(a) = \partial U/\partial a_i(a) \tag{10}$$

should exist and be equal and continuous, it is sufficient that for some $f(\delta)$ tending to zero with δ,

$$\int_{S(\delta)} |\rho| dV/r^{n-1} < f(\delta), \text{ for all } \delta, \tag{11}$$

where $r = |x - c|$, for all c in some sphere about a. Here $S(\delta)$ denotes the sphere with radius δ and centre c.

Explanation. In view of Theorem 2 equation (11) is simply a technical way of saying that the force integrals (3) converge *uniformly* in an open set containing a. It follows that $F_i(b)$ are continuous near a.

Proof. Without loss of generality, we can take $i = 1$ and suppose a is the origin. Then, writing $b = (h, 0, \ldots, 0)$, we define

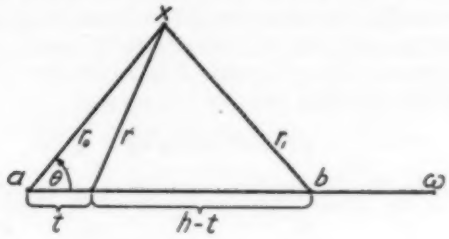
$$Q(h) = [U(b) - U(a)]\ /h - F_1(a).$$

The existence of all terms of $Q(h)$ for b near a follows directly from (11), which

implies (9) and the finiteness of (2) and hence the existence of $U(a)$, $U(b)$, $F_1(a)$. Our problem is to show that $Q(h) \to 0$ as $h \to 0$. But if $n > 3$

$$U(b) - U(a) = \int_R [r_1^{2-n} - r_0^{2-n}]\, \rho dV$$

where $r_0 = |x - a|$ and $r_1 = |x - b|$ as in the figure.



Substituting also for $F_1(a)$ from (3), we get

$$Q(h) = \int_R I(x, h)\rho dV, \text{ where } I(x, h) = \frac{1}{h}\left[\frac{1}{r_1^{n-2}} - \frac{1}{r_0^{n-2}}\right] - \frac{(n-2)x_1}{r_0^n}.$$

We now estimate the two preceding terms separately. We have

$$-\frac{1}{h}\left[\frac{1}{r_0^{n-2}} - \frac{1}{r_1^{n-2}}\right] = \frac{(r_0 - r_1)\sum_{k=0}^{n-3} r_0^k r_1^{n-3-k}}{h(r_0 r_1)(r_0 r_1)^{n-3}}.$$

But $\quad |r_0 - r_1| \leq |h|$, and $r_0^k r_1^{n-3-k} \leqq (r_0^{n-3} + r_1^{n-3})$.

Substituting and adding up the $n - 2$ terms, we get

$$\left|\frac{1}{h}\left[\frac{1}{r_0^{n-2}} - \frac{1}{r_1^{n-2}}\right]\right| \leqq (n-2)\left[\frac{1}{r_0^{n-2} r_1} + \frac{1}{r_0 r_1^{n-2}}\right].$$

Finally, since $1/r_0^{n-2}r_1 + 1/r_0 r_1^{n-2} \leqq r_0^{1-n} + r_1^{1-n}$ and $|x_1/r_0| \leqq 1$, we get the inequality

$$(*) \qquad |I(x, h)| \leqq (n-2)\left|\frac{2}{r_0^{n-1}} + \frac{1}{r_1^{n-1}}\right|.$$

Hence, using Lebesgue integrals, if $|h| < \delta$, and $S(\delta)$ and $S(b, h)$ denote spheres with centres a, b and radii δ, h respectively, we have

$$\int_{S(\delta)} |I(x, h)\rho|\, dV = (n-2)\left[2\int_{S(\delta)} \frac{|\rho| dV}{r_0^{n-1}} + \int_{S(b,h)} \frac{|\rho| dV}{r_1^{n-1}}\right] \leqq 3(n-2)f(2\delta),$$

since $S(2\delta)$ contains $S(b, h)$ if $|h| < \delta$. For this we require only that (11) hold along the x_1-axis to make the contribution to $Q(h)$ arbitrarily small provided δ is small enough and $|h| < \delta$.

But if h is small enough, the contribution from outside $S(\delta)$ is arbitrarily small. For we have, by Taylor's Theorem with remainder for the function r^{2-n} considered as a function of h at $h = 0$,

$$|I(x, h)| \leqq \tfrac{1}{2} h \sup_{0<t<h} \left| \partial^2(r^{2-n})/\partial t^2 \right| \leqq \tfrac{1}{2} hK(\delta),$$

since, if $h < \delta/2$, the second derivative is bounded outside $S(\delta)$. Hence the total contribution to $Q(h)$ from outside $S(\delta)$ is bounded by $\frac{1}{2}hK(\delta)C$, where $C = \int |\rho| dV$ as before, and can be made arbitrarily small. Adding together the contributions from inside and outside $S(\delta)$, we see that $Q(h) \to 0$ as $h \to 0$.

In case $n = 2$, $U(a)$ and $U(b)$ exist by (11) because

$$U = \int_R - \rho \log r dV \text{ and } |\rho \log r| \leq \frac{|\rho|}{r}.$$

Also, we have $-I(x, h) = \dfrac{1}{h}[\log r_1 - \log r_0] + \dfrac{x_1}{r_0^2}$. Now $|\log r_1 - \log r_0|$ $\leq |r_1 - r_0|\left[\dfrac{1}{r_1} + \dfrac{1}{r_0}\right]$ from the theorem of the mean; hence

$$|I(x, h)| \leq \frac{1}{r_1} + \frac{2}{r_0}, \text{ and } \int_{S(\delta)} |I(x, h, \rho)| dV = 0(\delta) \text{ if } |h| < \text{ as before.}$$

COROLLARY 1. *We have* (10) *provided uniformly*

$$|\rho(x)| \leqq g(r), \text{ where } \int_0^1 g(r)dr < \infty. \tag{11'}$$

COROLLARY 2. *We have* (10) *provided* $\rho(x) = 0(r_0^{\gamma-1})$ for $\gamma < 0$ near a.

Proof. The inequality (11) follows from the hypothesis, because for c sufficiently near a

$$\int_{S(\delta)} \frac{|\rho|\, dV}{r^{n-1}} \leq \int_{S(\delta)} \frac{AdV}{r_0^{1-\gamma} r^{n-1}} \leq \int_{S(\delta)} \frac{AdV}{r_0^{n-\gamma}} + \int_{S(\delta)} \frac{AdV}{r^{n-\gamma}} \leq \int_{S(\delta)} \frac{2AdV}{r^{n-\gamma}}$$

by [1], p. 148, Lemma III (a) generalized to n dimensions, and the last integral approaches 0 as $\delta \to 0$ uniformly in c.

Further inspection shows that for (10) to hold at a, it is sufficient for (11) to hold along the coordinate axes.

Clearly $F_1(x) = \lim_{h \to 0} \left[\dfrac{U(x + h) - U(x)}{h}\right]$ exists as a uniform limit for x in the sphere about a, and hence $F_1(x)$ is continuous, and similarly all $F_i(x)$ are continuous in the hyper-sphere about a. Hence N is continuous in Theorem 1, and we have

COROLLARY 3. *If* $\rho(x)$ *is continuous a.e., and* (11) *or* (11') *is valid for all* a *in* S, *then Gauss' Integral Theorem* (5) *is valid in the sense of improper Riemann integration. Thus it is sufficient that* $\rho(x)$ *be continuous.*

We shall now prove a somewhat weaker result under more general hypotheses.

THEOREM 4. *For almost every direction* $w = (w_1, \ldots, w_n)$, *the directional derivative* $\partial U/\partial w(a)$ *exists and satisfies*

$$\partial U/\partial w(a) = w_1F_1(a) + \ldots + w_nF_n(a), \tag{12}$$

provided that (9) *holds at* a; *i.e., provided the* $F_i(a)$ *are defined.*[11]

Explanation. We mean that the set of unit vectors w for which (12) fails, has spherical measure zero on the unit sphere Ω in n-space.

Proof. By Theorem 2, the right-hand side of (12) and $U(O)$ are defined. Moreover without losing generality, we can set $a = O$, and restrict our attention to an arbitrarily small sphere $S(\delta)$ with centre a, as in Theorem 3. Also as in Theorem 3, we can define (cf. the figure)

$$Q(h, w) = \frac{1}{h}[U(hw) - U(0)] - \sum_{i=1}^{n} w_iF_i(a)$$

$$= \int_{S(\delta)} I(x, h, w)\rho dV, \text{ where, if } n \geqq 3,$$

$$I(x, h, w) = \frac{1}{h}\left[\frac{1}{r_1^{n-2}} - \frac{1}{r_0^{n-2}}\right] - (n-2)\sum_{i=1}^{n} w_ix_i/r_0^n.$$

Moreover $U(hw)$ will exist if $\int_{S(\delta)} I(x, h, w)\rho dV$ exists. Further, we define the new function

$$J(x, h, w) = \sup_{0 < t \leqq h} |I(x, t, w)|.$$

Clearly $I(x, t, w)$ is continuous, except when x is in the direction w. Hence ([4], p. 42) for fixed h, $J(x, h, w)$ is *lower semicontinuous* in the product-space $\Omega \times S(\delta)$ of couples (w, x), except on the set of $(2n-1)$-dimensional measure zero where x is in the direction w. Hence ([4], p. 43) $J(x, h, w)$ is *measurable* on $\Omega \times S(\delta)$, and we can define the multiple integral (J is non-negative)

$$Q_1 = \frac{1}{\omega_n}\int_\Omega \int_{S(\delta)} J(x, h, w)\,|\rho|\,dVd\Omega \leqq +\infty. \tag{13}$$

But by the inequality (*) of the proof of Theorem 3,

$$J(x, h, w) \leqq (n-2)\left[\frac{2}{r_0^{n-1}} + \left(\frac{1}{r^{n-1}}\right)\max\right].$$

While by trigonometry, $r \geqq r_0 \sin\theta = \theta r_0/2\pi$, if θ is acute, while $r \geqq r_0 \geqq \theta r_0/2\pi$ if θ is obtuse (cf. fig.). Substituting we get

$$Q_1 = \frac{n-2}{\omega_n}\int_\Omega d\Omega \int_{S(\delta)}\left[\frac{2}{r_0^{n-1}} + \frac{B}{r_0^{n-1}\theta^{n-1}}\right]|\rho|\,dV,$$

where the use of the iterated integral is permissible by the Fubini Theorem

[11] This result is closely related to the discussion of G. C. Evans, *Trans. Amer. Math. Soc.*, vol. 37 (1935), 234.

([4], p. 82) provided the right-hand integral is finite. But $d\Omega \leqq \omega_{n-1}\theta^{n-1}d\theta$ by spherical geometry; hence by (9′)

$$Q_1 \leqq 2(n-2)g(\delta) + \frac{(n-2)\omega_{n-1}}{\omega_n}\int_{S(\delta)} |\rho|\, dV \int_\Omega \frac{B}{r_0^{n-1}\theta^{n-1}}\,\theta^{n-1}d\theta$$
$$= (n-2)\,[-2 + \pi B\omega_{n-1}/\omega_n]\, g(\delta).$$

But by (9′), for δ sufficiently small and all $h < \delta$, this can be made arbitrarily small; hence it is finite.

By the Fubini Theorem and (13),

$$\int_\Omega d\Omega \int_{S(\delta)} J(x, h, w)\, |\rho|\, dV = Q_1\omega_n$$

is arbitrarily small if δ is sufficiently small. Hence the integrand $\int_{S(\delta)} J(x, h, w) \cdot |\rho|\, dV$ can be made less than ϵ except on a set of directions of Ω-measure at most ϵ. Hence

$$\sup_{0<t<h} |Q(t, w)| \leqq \int_{S(\delta)} J(x, h, w,)\, |\rho|\, dV < \epsilon$$

except on the same set of directions. This immediately implies (12), including the existence of $\partial U/\partial w(a)$, for almost every direction. If $n = 2$, $|I(x, h)| \leq \frac{1}{r} + \frac{2}{r_0}$, and the proof follows as above.

COROLLARY 1. *Theorem* 4 *holds if* (9) *is replaced by* $\rho = 0\left(\frac{1}{r_0^{1-\gamma}}\right)$ *or even* $\rho = 0\left(\frac{1}{r_0|\log r_0|^{1+\gamma}}\right)$ *for* r_0 *sufficiently small.*

It should be noted that in Theorems 3 and 4 dealing with the first partial derivatives of U no mention is made of the Hölder condition on ρ, namely that there exist A and $\alpha > 0$ such that $P(x) \leqq Ar^\alpha$ if $|x| = r \leqq r_0$. (See footnote 3.)

4. Poisson equation. We now extend Petrini's result on the existence of the "generalized Laplacian"

$$\nabla_P^2\, U(a) = \lim_{h\to 0} \frac{1}{h^2} \sum_{i=1}^{n} [U(a + he_i) + U(a - he_i) - 2U(a)], \tag{14}$$

defined by second differences, instead of second derivatives. Define $\rho(x)$ to be *mean continuous* at a, if and only if

$$\epsilon(c) = \int_{S(c)} |\rho - \rho_0|\, dV/c^n \to 0 \text{ as } c \to 0. \tag{15}$$

If ρ is bounded, this means that ρ differs from ρ_0 by less than $\epsilon(c)$ except on a set of arbitrarily small relative measure, in sufficiently small spheres $S(c)$ with radius c and centre a.

THEOREM 5. *Let* ρ *be measurable and bounded near* a, *and "mean continuous" at* a. *Then the generalized Laplacian* (14) *exists and satisfies the Poisson equation* (1) *with* $\rho = \rho_0$.

Remark. If ρ is continuous, then it is bounded and measurable, and (15) holds. Hence Theorem 5 contains the result of Petrini; moreover measurability, and continuity at a are sufficient to imply our result.

Proof. For any $h > 0$, the sum in square brackets in (14) as defined by (2) exists, since ρ is bounded and measurable, whence ρr^{2-n} is integrable. Again, the case of constant density is covered by elementary formulas; hence we can suppose $\rho_0 = 0$ without loss of generality; this we now do.

If we represent each term of (14) by (2), and sum under the integral sign, the right-hand quantity in (14) becomes

$$(16) \qquad \frac{1}{h^2}\int J(x;h)\rho(x)dV, \text{ where}$$

$$J(x,h) = \sum_{i=1}^{n}\left[\frac{1}{|a+he_i-x|^{n-2}} + \frac{1}{|a-he_i-x|^{n-2}} - \frac{2}{|a-x|^{n-2}}\right].$$

We now divide the definite integral of (16) into the contribution $I_1(h)$ from inside a sphere of centre a and radius $8h$, and the contributions $I_k(h)$ from the spherical shells $S(h,k)$: $2^k h \leqq |x-a| \leqq 2^{k+1}h$ $[k = 3, 4, 5, \ldots]$. We write $|x-a| = r$.

In $S(h,k)$, expanding $|x-y|^{2-n}$ by Taylor's Theorem with remainder out to third partial derivatives, we get $|J(x,h)| \leqq Kh^3/r^{n+1} \leqq K'h^3/2^{(n+1)k}h^{n+1}$. Also, $\int_{S(h,k)} |\rho|\,dV$ is at most $(2^{k+1}h)^n\epsilon(2^{k+1}h)$. Hence $|I_k(h)| = h^3K^*\epsilon(2^{k+1}h)2^{-k}$, where K^* is an absolute constant. Hence by (15),

$$\lim_{h\to 0}\frac{1}{h^3}\sum_{k=3}^{\infty} I_k(h) \leqq K^* \lim_{h\to 0}\sum_{k=3}^{\infty} 2^{-k}\epsilon(2^{k+1}h) = 0.$$

It remains to show that $\lim_{h\to 0} I_1(h)/h^2 = 0$. Breaking up $J(x,h)$ into its $3n$ summands, it is clearly sufficient to show that whether b is a, $a + he_i$, or $a - he_i$, we have

$$(17) \qquad \lim_{h\to 0}\frac{1}{h^2}\int_{S(9h)} r'^{2-n}|\rho(x)|\,dV = 0 \qquad [r' = |x-b|].$$

Here $S(9h)$ denotes the sphere with centre b and radius $9h$.

But since ρ is bounded, $|\rho_i| \leqq M$, and so for any $\delta > 0$,

$$\frac{1}{h^2}\int_{S(\delta)} \frac{|\rho(x)|dV}{r'^{n-2}} \leqq \frac{1}{h^2}\int_{S(\delta)} Mr'^{2-n}r'^{n-1}dr \leqq 41\,M\delta^2/h^2.$$

Also,

$$\frac{1}{h^2}\int_{r'=\delta}^{9h} r'^{2-n}|\rho|dV \leqq \frac{1}{h^2}\delta^{2-n}\int_{r'=\delta}^{9h}|\rho|dV$$

$$\leqq (10h)^n\epsilon(10h)/h^2\delta^{n-2} \leqq K^*(h/\delta)^{n-2}\epsilon(10h).$$

By choosing $\delta/h = \sqrt[n]{\epsilon(10h)}$, we infer (17). — Note that we used $\epsilon(10h)$ and not $\epsilon(9h)$, because (15) refers to spheres with centre a and not spheres with centre b.

COROLLARY 1. *Under the preceding hypothesis, using the notation of* (7), $\lim_{c \to 0} [\overline{U}(S) - U(O)]/c^2$ *exists and is* $\omega_n \nabla_\rho^2 U(O)$.

Proof. We average with respect to all choices of axes, and observe that the inequalities above hold for *all* choices of axes.

COROLLARY 2. *If* $\nabla^2 U$ *exists and* $\rho(x)$ *satisfies the preceding hypotheses, then* (1) *holds.*

Proof. Expanding in Taylor's series with a remainder $o(h^2)$, the existence of $\nabla^2 U$ implies that $\nabla_\rho{}^2 U$ exists and has the same value. It is of course known that even the continuity everywhere of ρ need not imply the existence of $\nabla^2 U$.

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Harvard University